

## Modular Invariants and Subfactors

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*This contribution is dedicated to Sergio Doplicher and John E. Roberts  
on the occasion of their 60th birthdays.*

**Abstract.** In this lecture we explain the intimate relationship between modular invariants in conformal field theory and braided subfactors in operator algebras. Our analysis is based on an approach to modular invariants using braided sector induction (“ $\alpha$ -induction”) arising from the treatment of conformal field theory in the Doplicher-Haag-Roberts framework. Many properties of modular invariants which have so far been noticed empirically and considered mysterious can be rigorously derived in a very general setting in the subfactor context. For example, the connection between modular invariants and graphs (cf. the A-D-E classification for  $SU(2)_k$ ) finds a natural explanation and interpretation. We try to give an overview on the current state of affairs concerning the expected equivalence between the classifications of braided subfactors and modular invariant two-dimensional conformal field theories.

### 1 Modular invariants in rational conformal field theory

It is common knowledge that many, possibly all, models in rational conformal field theory (RCFT) are related to current algebras (or “WZW models”). A crucial role in the analysis of such current algebra models is played by their representation theory. In more mathematical terms: One has to study the unitary integrable

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highest weight modules over affine Lie algebras (cf. [37, 23]), or, if you prefer the “exponentiated version”, the positive energy representations of loop groups (cf. [51]), as e.g.  $LSU(n)$ . The positive energy representations are labelled by a level  $k$ , a positive integer, and by weights  $\lambda$  in the corresponding Weyl alcove. Among these, there is a distinguished representation, the “vacuum representation” associated to the weight  $\lambda = 0$ . For each positive energy representation  $\pi_\lambda$ , acting on a Hilbert space  $H_\lambda$ , we can define a (specialized) character

$$\chi_\lambda(\tau) = \text{tr}_{H_\lambda} \exp(2\pi i \tau L_0),$$

where  $\tau \in \mathbb{C}$  is in the upper half plane, and  $L_0$  denotes the conformal energy operator which generates the rotations on the unit circle  $S^1$  and which arises from the affine Lie algebra by the Sugawara construction. Its lowest eigenvalues  $h_\lambda \geq 0$  is called “conformal dimension”, and the vacuum representation has the unique conformal dimension  $h_0 = 0$ . (More generally, the un-specialized characters are defined by using in addition other variables corresponding to Cartan subalgebra generators which we omit here for the sake of simplicity.) It is an important and fascinating fact that the characters are modular functions. More precisely, at each fixed level  $k$  there are unitary matrices  $S$  and  $T$ , the Kac-Peterson matrices (see [37]), such that

$$\chi_\lambda(-1/\tau) = \sum_\mu S_{\lambda,\mu} \chi_\mu(\tau), \quad \chi_\lambda(\tau+1) = \sum_\mu T_{\lambda,\mu} \chi_\mu(\tau).$$

Thus there is an action of the modular group  $SL(2;\mathbb{Z})$  on the upper half plane variable  $\tau$  with generators  $\mathcal{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ :  $\tau \mapsto -1/\tau$ , and  $\mathcal{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ :  $\tau \mapsto \tau+1$ , transforming the family of characters  $\{\chi_\lambda\}$  unitarily. Besides the modular relation

$$(ST)^3 = S^2 \tag{1.1}$$

these matrices have some remarkable properties. In particular,  $T$  is diagonal, and the diagonal entries are up to an overall normalizing factor given by phases  $\exp(2\pi i h_\lambda)$ ,  $S$  is symmetric,  $S^2 \equiv C$  is a permutation matrix expressing “charge conjugation” which leaves conformal dimensions invariant since  $CT = TC$ , and  $S_{\lambda,0} \geq S_{0,0} > 0$ . Moreover,  $S$  produces non-negative integers  $N_{\lambda,\mu}^\nu$  by the Verlinde formula [62],

$$N_{\lambda,\mu}^\nu = \sum_\rho \frac{S_{\lambda,\rho} S_{\mu,\rho} S_{\nu,\rho}^*}{S_{0,\rho}}, \tag{1.2}$$

and these integers, called “fusion rules”, define a commutative “fusion rule algebra”. Combining Eq. (1.1) and Eq. (1.2) yields

$$S_{\lambda,\mu} = S_{0,0} \sum_\rho \exp(2\pi i(h_\lambda + h_\mu - h_\rho)) N_{\lambda,\mu}^\rho d_\rho,$$

where  $S_{0,0} = 1/\sqrt{w}$  with  $w = \sum_\lambda d_\lambda^2$  and “quantum dimensions”  $d_\lambda = S_{\lambda,0}/S_{0,0}$ .

In order to tackle the classification problem of RCFT, one in particular tries to find all two-dimensional conformal field theories which contain two copies of the chiral current algebra as left and right movers on the (compactified) light rays, and such that the Hilbert space of the 2D theory decomposes upon restriction to the tensor product of chiral algebras into a direct sum of tensor products of positive energy representations according to

$$\mathcal{H}_{\text{phys}} = \bigoplus_{\lambda,\mu} Z_{\lambda,\mu} H_\lambda \otimes H_\mu, \tag{1.3}$$

with non-negative integer multiplicities

$$Z_{\lambda,\mu} = 0, 1, 2, \dots, \quad \text{and} \quad Z_{0,0} = 1. \quad (1.4)$$

The latter normalization condition expresses the uniqueness of the vacuum.

In fact, though there is no direct equivalence, there is a deep connection between this classification problem and the problem of classifying modular invariant partition functions

$$\mathcal{Z}(\tau) = \sum_{\lambda,\mu} Z_{\lambda,\mu} \chi_\lambda(\tau) \chi_\mu(\tau)^* \quad (1.5)$$

such that  $\mathcal{Z}(-1/\tau) = \mathcal{Z}(\tau) = \mathcal{Z}(\tau + 1)$ . Modular invariant partition functions arise as continuum limits in statistical mechanics and play a fundamental role in conformal field theory. The classification problem for modular invariant partition functions seems to be more handy than the classification of 2D RCFT's as it will only require solutions to the matrix equations  $SZ = ZS$ ,  $TZ = ZT$ , subject to the constraints of Eq. (1.4). In fact, as noticed by Gannon [25], for given matrices  $S$  and  $T$  as above, there are only finitely many solutions since  $w = S_{0,0}^{-2}$  is an overall bound for the sum of all entries of  $Z$ . There is also an inequality

$$Z_{\lambda,\mu} \leq d_\lambda d_\mu \quad (1.6)$$

for each individual entry of a modular invariant coupling matrix [9]. So clearly for a fixed model, e.g. for a certain  $SU(n)$  at a fixed level  $k$ , there is only a finite number of modular invariants. However, ambitious scientists are usually interested in classifying modular invariants for entire series of models, e.g.  $SU(n)$ , fixed rank but all levels. Unfortunately, this ambition turned out to meet hard problems, and complete classifications exist only for  $n = 2$ , the celebrated A-D-E classification of [12, 13, 38], and for  $n = 3$  [26]. Gannon has recently informed us that he has completed the  $SU(4)$  case up to levels  $k \leq 5,000$  and with similar bounds also other Lie groups, but otherwise there are still many open problems around. (At fixed low levels  $k < 4$ , there are however classifications for all  $SU(n)$ , see [27].) Anyway, the classification problem of modular invariants is definitely a fascinating area of mathematical physics with many deep connections to other branches of mathematics and physics, see e.g. [29].

To be a bit more illustrative, let us see some examples. For  $SU(2)$  at level  $k$ , the positive energy representations are labelled simply by spins  $\lambda = 0, 1, 2, \dots, k$ . At level  $k = 6$ , there are two modular invariants, which read, when written in the form of Eq. (1.5),

$$\begin{aligned} \mathcal{Z}_{A_7} &= |\chi_0|^2 + |\chi_1|^2 + |\chi_2|^2 + |\chi_3|^2 + |\chi_4|^2 + |\chi_5|^2 + |\chi_6|^2, \\ \mathcal{Z}_{D_5} &= |\chi_0|^2 + |\chi_2|^2 + |\chi_4|^2 + |\chi_6|^2 + \chi_1 \chi_5^* + \chi_3 \chi_3^* + \chi_5 \chi_1^*. \end{aligned}$$

At level  $k = 16$  there are three:

$$\begin{aligned} \mathcal{Z}_{A_{17}} &= \sum_{\lambda=0}^{16} |\chi_\lambda|^2 \\ \mathcal{Z}_{D_{10}} &= |\chi_0 + \chi_{16}|^2 + |\chi_2 + \chi_{14}|^2 + |\chi_4 + \chi_{12}|^2 + |\chi_6 + \chi_{10}|^2 + 2|\chi_8|^2, \\ \mathcal{Z}_{E_7} &= |\chi_0 + \chi_{16}|^2 + |\chi_4 + \chi_{12}|^2 + |\chi_6 + \chi_{10}|^2 + |\chi_8|^2 \\ &\quad + (\chi_2 + \chi_{14})\chi_8^* + \chi_8(\chi_2 + \chi_{14})^*. \end{aligned}$$

There is so much structure visible already in these relatively simple examples that we would like to comment on this. Let us first explain the labelling of the  $\mathcal{Z}$ 's by A-D-E Dynkin diagrams. It was noticed in [12] that the diagonal terms of each invariant appearing at level  $k$  reproduce exactly the Coxeter exponents of one of the Dynkin diagrams with Coxeter number  $k + 2$ , and this amounts to a bijective

correspondence between all the  $SU(2)_k$ ,  $k = 1, 2, \dots$ , modular invariants (The list of [12] was proven to be complete in [13, 38], and for a more recent elegant proof see [28]) and all the A-D-E Dynkin diagrams. More precisely, the labelling the invariants by Dynkin diagrams is such that the diagonal entry  $Z_{\rho,\rho}$  of the invariant associated to the A-D-E diagram  $G_1$  is exactly the multiplicity of the eigenvalue

$$\frac{S_{1,\rho}}{S_{0,\rho}} = 2 \cos \left( \frac{(\rho+1)\pi}{k+2} \right)$$

of the (adjacency matrix of the) graph  $G_1$ . This empirical observation was certainly awaiting a good explanation! Nahm found [47] a systematic relation between the diagonal part of the  $SU(2)$  invariants and Lie algebra exponents using quaternionic coset spaces. But Nahm's ideas do not explain that the A-D-E observation turned out to be just the “ $SU(2)$ -spin-1-tip of the iceberg”: First of all, the A-D-E diagrams  $G_1$  are just the spin-1 member of a family of graphs  $G_\lambda$ ,  $\lambda = 0, 1, 2, \dots, k$ , which form a non-negative integer valued matrix representation (nimrep, for short) of the  $SU(2)_k$  Verlinde fusion rules:

$$G_\lambda G_\mu = \sum_\nu N_{\lambda,\mu}^\nu G_\nu,$$

with eigenvalue multiplicities  $\text{mult}_{G_\lambda}(S_{\lambda,\rho}/S_{0,\rho}) = Z_{\rho,\rho}$ . (For the Dynkin diagrams of type A, the  $G_\lambda$ 's are just the fusion matrices, and the appearance of each character  $\gamma_\rho(\cdot) = S_{\cdot,\rho}/S_{0,\rho}$  with multiplicity  $Z_{\rho,\rho} = 1$  for all  $\rho$  is just the Verlinde formula.) And even more, guided by the observations for  $SU(2)$ , Di Francesco and Zuber found [16, 17, 15] (see also related work [49, 2]) that there are graphs and nimreps for  $SU(3)$  and also higher rank  $SU(n)$  modular invariants which fall into the analogous scheme, i.e. such that you just have to replace the labels (“weights”), fusion rules and S-matrices by the corresponding  $SU(n)$  data. In the subfactor context, we will see that these nimreps arise from a certain braided sector induction, called “ $\alpha$ -induction”; the matrix entries are non-negative integers since they are dimensions of certain intertwiner spaces, the representation property is due to the fact that  $\alpha$ -induction preserves fusion rules, and a general theorem determining the character multiplicities to be given by the diagonal entries of the modular invariant coupling matrix can be proven.

Next we address the distinction of type I and type II modular invariants. Note that, whatever model we are looking at, there will always be at least one solution, the diagonal partition function

$$\mathcal{Z} = \sum_\lambda |\chi_\lambda|^2,$$

which is always modular invariant, equivalently expressed in the fact that the unit matrix,  $Z_{\lambda,\mu} = \delta_{\lambda,\mu}$ , always commutes with  $S$  and  $T$ . For  $SU(2)$ , these are the invariants labelled by Dynkin diagrams  $A_{k+1}$ . More generally, there may be permutation invariants

$$\mathcal{Z} = \sum_\lambda \chi_\lambda \chi_{\vartheta(\lambda)}^*,$$

whenever  $\vartheta$  is a permutation of the labels which preserves the fusion rules, the vacuum, and the conformal dimensions modular integers. The above displayed  $D_5$  invariant for  $SU(2)_6$  is an example for such an automorphism. In general the charge conjugation matrix  $C$  is also such an automorphism — which is however trivial in the special case of  $SU(2)$ . Moore and Seiberg argue in [46] (see also [18]) that after a “maximal extension of the chiral algebra” (the hardest part is to make this mathematically precise) the partition function of a RCFT is at most a permutation

matrix  $Z_{\tau, \tau'}^{\text{ext}} = \delta_{\tau, \vartheta(\tau')}$ , where  $\tau, \tau'$  now label the representations of the extended chiral algebra and  $\vartheta$  denotes a permutation of these with analogous invariance properties. Decomposing the extended characters  $\chi_{\tau}^{\text{ext}}$  in terms of the original characters  $\chi_{\lambda}$ , we have  $\chi_{\tau}^{\text{ext}} = \sum_{\lambda} b_{\tau, \lambda} \chi_{\lambda}$  for some non-negative integral branching coefficients  $b_{\tau, \lambda}$ . The maximal extension yields the coupling matrix expression

$$Z_{\lambda, \mu} = \sum_{\tau} b_{\tau, \lambda} b_{\vartheta(\tau), \mu}. \quad (1.7)$$

Di Francesco and Zuber [17] called invariants which arise from the diagonal invariant of the maximal extension (i.e. for which  $\vartheta$  is trivial) “type I”, and invariants corresponding to non-trivial automorphisms of the extended fusion rules were called “type II”. Looking at the above displayed  $SU(2)_6$  and  $SU(2)_{16}$  invariants, we find for example that  $\mathcal{Z}_{A_7}$ ,  $\mathcal{Z}_{A_{17}}$  and  $\mathcal{Z}_{D_{10}}$  are type I whereas  $\mathcal{Z}_{D_5}$  and  $\mathcal{Z}_{E_7}$  are type II.

Let us finally remark that the A-D-E classification generalizes in some sense to the entire classification problem of modular invariants in RCFT. The class of diagonal modular invariants and their conjugations is often denoted by  $\mathcal{A}$ . A wider class is given by the “simple current invariants” (see e.g. [59, 60, 30]) for which  $Z_{\lambda, \mu} \neq 0$  implies  $N_{\sigma, \lambda}^{\mu} = 1$  for some simple current  $\sigma$  (i.e. a label with  $d_{\sigma} = 1$ ). The class of simple current invariants minus the  $\mathcal{A}$  class is often denoted by  $\mathcal{D}$ . The remaining modular invariants, which are typically relatively few, are called “exceptionals” and their class is denoted by  $\mathcal{E}$ . In fact, considering a loop group model (with fixed rank) at all levels, the  $\mathcal{A}$  and, if there are simple currents,  $\mathcal{D}$  classes give infinite, very well-behaving series of invariants and there is only a finite number of exceptionals. The graphs which have been associated to diagonal  $\mathcal{A}$  invariants are basically the Weyl alcove with edges corresponding to the fusion with the fundamental generator. The graphs which have been associated to simple current  $\mathcal{D}$  invariants are simply orbifolds of the  $\mathcal{A}$ -type graphs with respect to a cyclic simple current symmetry. (E.g. the Dynkin diagrams  $D_{\varrho+2}$  are  $\mathbb{Z}_2$  orbifolds of the  $A_{2\varrho+1}$  graphs at even levels  $k = 2\varrho$ ,  $\varrho = 2, 3, \dots$ ) The graphs associated to  $\mathcal{E}$  invariants may be considered as orbifolds with respect to a more subtle, non-group-like symmetry. In any case, it was noticed (see [15]) that one can associate intrinsic fusion rules algebras to the graphs exactly in the type I cases, and that they contain fusion subalgebras corresponding to the Verlinde fusion rules of the extended characters. All this finds a natural explanation in the subfactor framework.

## 2 Modular invariants from subfactors through $\alpha$ -induction

Now let us switch to a different topic: Subfactors. (See [19] as a general reference on subfactors.) At the first sight, this topic does not seem to have anything to do with modular invariants. However, soon we shall see that it does, and in fact quite a lot!

Let  $A$  and  $B$  be type III von Neumann factors. A unital  $*$ -homomorphism  $\rho : A \rightarrow B$  is called a  $B$ - $A$  morphism. The positive number  $d_{\rho} = [B : \rho(A)]^{1/2}$  is called the statistical dimension of  $\rho$ ; here  $[B : \rho(A)]$  is the Jones-Kosaki index [36, 39] of the subfactor  $\rho(A) \subset B$ . Some  $B$ - $A$  morphism  $\rho'$  is called equivalent to  $\rho$  if  $\rho' = \text{Ad}(u) \circ \rho$  for some unitary  $u \in B$ . The equivalence class  $[\rho]$  of  $\rho$  is called the  $B$ - $A$  sector of  $\rho$ . If  $\rho$  and  $\sigma$  are  $B$ - $A$  morphisms with finite statistical dimensions, then the vector space of intertwiners

$$\text{Hom}(\rho, \sigma) = \{t \in B : t\rho(a) = \sigma(a)t, a \in A\}$$

is finite-dimensional, and we denote its dimension by  $\langle \rho, \sigma \rangle$ . In fact  $\langle \rho, \sigma \rangle \leq d_{\rho} d_{\sigma}$ . A  $B$ - $A$  morphism is called irreducible if  $\langle \rho, \rho \rangle = 1$ , i.e. if  $\text{Hom}(\rho, \rho) = \mathbb{C} \mathbf{1}_B$ . Then,

if  $\langle \rho, \tau \rangle \neq 0$  for some (possibly reducible)  $B$ - $A$  morphism  $\tau$ , then  $[\rho]$  is called an irreducible subsector of  $[\tau]$  with multiplicity  $\langle \rho, \tau \rangle$ . An irreducible  $A$ - $B$  morphism  $\bar{\rho}$  is a conjugate morphism of the irreducible  $\rho$  if and only if  $[\bar{\rho}\rho]$  contains the trivial sector  $[\text{id}_A]$  as a subsector, and then  $\langle \rho\bar{\rho}, \text{id}_B \rangle = 1 = \langle \bar{\rho}\rho, \text{id}_A \rangle$  automatically [34]. The map  $\phi_\rho : B \rightarrow A$ ,  $b \mapsto r_\rho^* \bar{\rho}(b) r_\rho$ , is called the (unique) standard left inverse and satisfies

$$\phi_\rho(\rho(a)b\rho(a')) = a\phi_\rho(b)a', \quad a, a' \in A, \quad b \in B.$$

Now let  $N$  be a type III factors. We assume that we have a given finite system of  $N$ - $N$  morphisms  ${}_N\mathcal{X}_N$ , i.e.  ${}_N\mathcal{X}_N \subset \text{End}(N)$  is finite and

- each  $\lambda \in {}_N\mathcal{X}_N$  is irreducible, i.e.  $\lambda(N)' \cap N = \mathbb{C}\mathbf{1}_N$ ,
- each  $\lambda \in {}_N\mathcal{X}_N$  has finite statistical dimension, i.e.  $d_\lambda = [N : \lambda(N)]^{1/2} < \infty$ ,
- the morphisms are pairwise inequivalent, i.e.  $\langle \lambda, \mu \rangle = 0$  whenever  $\lambda \neq \mu$ ,
- the identity morphism belongs to the system, i.e.  $\text{id} \in {}_N\mathcal{X}_N$ ,
- the system is closed under conjugation, i.e. for each  $\lambda \in {}_N\mathcal{X}_N$  there is some  $\bar{\lambda} \in {}_N\mathcal{X}_N$  such that  $\langle \bar{\lambda}\lambda, \text{id}_N \rangle = 1$ ,
- the system is closed under fusion, i.e. any composition  $[\lambda] \cdot [\mu] \equiv [\lambda \circ \mu]$ ,  $\lambda, \mu \in {}_N\mathcal{X}_N$  has only subsectors arising from  ${}_N\mathcal{X}_N$ ; we write  $[\lambda] \cdot [\mu] = \bigoplus_{\nu \in {}_N\mathcal{X}_N} \langle \lambda\mu, \nu \rangle [\nu]$ .

We now also assume that our system  ${}_N\mathcal{X}_N$  is braided: For any pair  $\lambda, \mu \in {}_N\mathcal{X}_N$  there is a unitary operator  $\varepsilon^+(\lambda, \mu) \in \text{Hom}(\lambda\mu, \mu\lambda)$  such that the braiding fusion equations,

$$\begin{aligned} \rho(t)\varepsilon^+(\lambda, \rho) &= \varepsilon^+(\mu, \rho)\mu(\varepsilon^+(\nu, \rho))t, \\ t\varepsilon^+(\rho, \lambda) &= \mu(\varepsilon^+(\rho, \nu))\varepsilon^+(\rho, \mu)\rho(t), \\ \rho(t)^*\varepsilon^+(\mu, \rho)\mu(\varepsilon^+(\nu, \rho)) &= \varepsilon^+(\lambda, \rho)t^*, \\ t^*\mu(\varepsilon^+(\rho, \nu))\varepsilon^+(\rho, \mu) &= \varepsilon^+(\rho, \lambda)\rho(t)^*, \end{aligned} \tag{2.1}$$

hold whenever  $\lambda, \mu, \nu \in {}_N\mathcal{X}_N$  and  $t \in \text{Hom}(\lambda, \mu\nu)$ . The unitaries  $\varepsilon^+(\lambda, \mu)$  are called statistics operators. Note that a braiding  $\varepsilon \equiv \varepsilon^+$  always comes along with another “opposite” braiding  $\varepsilon^-$ , namely operators  $\varepsilon^-(\lambda, \mu) = \varepsilon^+(\mu, \lambda)^*$ , satisfy the same relations. The unitaries  $\varepsilon^+(\lambda, \mu)$  and  $\varepsilon^-(\lambda, \mu)$  are different in general but may coincide for some  $\lambda, \mu$  — at least if one of them is the identity morphism since Eq. (2.1) implies  $\varepsilon^+(\lambda, \text{id}) = \mathbf{1}$ .

Exactly as in the DHR theory (cf. [33]),  $d_\lambda \phi_\lambda(\varepsilon^+(\lambda, \lambda)) \in \text{Hom}(\lambda, \lambda)$  is unitary, and so one defines the statistics phase  $\omega_\lambda \in \mathbb{T}$  by

$$d_\lambda \phi_\lambda(\varepsilon^+(\lambda, \lambda)) \in \text{Hom}(\lambda, \lambda) = \omega_\lambda \mathbf{1}.$$

Now consider the following matrices  $\Omega$  and  $Y$ ,

$$\begin{aligned} \Omega_{\lambda, \mu} &= \delta_{\lambda, \mu} \omega_\lambda, \\ Y_{\lambda, \mu} &= \sum_{\rho \in {}_N\mathcal{X}_N} \frac{\omega_\lambda \omega_\mu}{\omega_\rho} \langle \lambda\mu, \rho \rangle d_\rho, \end{aligned}$$

with indices labelled by  ${}_N\mathcal{X}_N$ , and we will use the label “0” for the identity morphism  $\text{id} \in {}_N\mathcal{X}_N$ . Then one checks that  $Y$  is symmetric, that  $Y_{\bar{\lambda}, \mu} = Y_{\lambda, \mu}^*$  as well as  $Y_{\lambda, 0} = d_\lambda$ . The  $Y$ - and  $\Omega$ -matrices obey  $\Omega Y \Omega Y \Omega = z Y$  where  $z = \sum_\lambda d_\lambda^2 \omega_\lambda$  [52, 21, 20]. If  $z \neq 0$  we put  $c = 4 \arg(z)/\pi$ , which is defined modulo 8, and call it the “central charge”, and then statistics S- and T-matrices defined by

$$S = |z|^{-1} Y, \quad T = e^{-i\pi c/12} \Omega$$

hence fulfill  $TSTST = S$ . With  $C$  denoting the sector conjugation matrix,  $C_{\lambda,\mu} = \delta_{\lambda,\bar{\mu}}$ , one finds  $CT = TC$  and  $CS = SC$ . Rehren showed in [52] that the following conditions are equivalent:

- The braiding  $\varepsilon$  is non-degenerate, i.e.  $\varepsilon^+(\lambda, \mu) = \varepsilon^-(\lambda, \mu)$  for all  $\mu \in {}_N\mathcal{X}_N$  only if  $\lambda = \text{id}$ .
- One has  $|z|^2 = w$  with the global index  $w = \sum_{\lambda} d_{\lambda}^2$  and  $S$  is unitary, so that  $S$  and  $T$  are indeed the standard generators  $\mathcal{S} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $\mathcal{T} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  in a unitary representation of  $SL(2; \mathbb{Z})$ , i.e. fulfill Eq. (1.1), and indeed  $S^2 = C$ . Moreover, a Verlinde formula holds (cf. Eq. (1.2)):

$$\sum_{\rho \in {}_N\mathcal{X}_N} \frac{S_{\lambda,\rho}}{S_{0,\rho}} S_{\mu,\rho} S_{\nu,\rho}^* = \langle \lambda\mu, \nu \rangle.$$

So here we have obtained another representation of the modular group  $SL(2; \mathbb{Z})$ , and in fact we seem to be dealing with precisely the same categorical structures as in RCFT (at least if the braiding is non-degenerate)!

Now let  $LG$  be a loop group (associated to a simple, simply connected loop group  $G$ ). Let  $L_I G$  denote the subgroup of loops which are trivial off some proper interval  $I \subset S^1$ . Then in each level  $k$  vacuum representation  $\pi_0$  of  $LG$ , we naturally obtain a net<sup>1</sup> of type III factors  $\{N(I)\}$  indexed by proper intervals  $I \subset S^1$  by taking  $N(I) = \pi_0(L_I G)''$  (see [65, 22, 1]). Since the DHR selection criterion (cf. [33]) is met in the (level  $k$ ) positive energy representations  $\pi_{\lambda}$ , there are DHR endomorphisms  $\lambda$  naturally associated with them. (By some abuse of notation we use the same symbols for labels and endomorphisms.) Now it is very natural to expect the following (a conjecture which actually goes beyond loop groups, see e.g. [22]): We anticipate that the statistics phases are the exponentiated conformal dimensions, i.e. that

$$\omega_{\lambda} = \exp(2\pi i h_{\lambda}), \quad (2.2)$$

and that the RCFT Verlinde fusion coincides with the (DHR superselection) sector fusion, i.e. that

$$N_{\lambda,\mu}^{\nu} = \langle \lambda\mu, \nu \rangle. \quad (2.3)$$

(And in turn that the RCFT quantum dimensions equal the statistical dimensions.) In other words, we expect that the normalized matrices  $Y$  and  $\Omega$ , namely the statistics S- and T-matrices are identical with the Kac-Peterson S- and T-modular matrices which perform the conformal character transformations. Fortunately general results have been proven [21, 20, 32] for the “conformal spins statistics theorem”, Eq. (2.2). For Eq. (2.3) there are proofs available [65, 41, 61, 3, 4] unfortunately only for special models.<sup>2</sup> Anyway, we will be mainly concerned with  $SU(n)_k$  here, so we can take the equality of statistics and Kac-Peterson matrices for granted, thanks to [65].

To summarize the above paragraphs, we have seen that a factor with a (non-degenerately braided system of endomorphisms gives rise to a unitary representation of the modular group  $SL(2; \mathbb{Z})$  via matrices  $S$  and  $T$  which are analogous to the Kac-Peterson matrices in RCFT. So what about modular invariants? As we shall see, modular invariants appear naturally in the operator algebraic setting when

<sup>1</sup>In fact a proper net is obtained only if we remove a “point at infinity” from the circle  $S^1$ .

<sup>2</sup>Antony Wassermann has informed us that he has extended his results for  $SU(n)_k$  fusion [65] to all simple, simply connected loop groups; and with Toledano-Laredo all but  $E_8$  using a variant of the Dotsenko-Fateev differential equation considered in his thesis [61].

we consider *subfactors* with a braiding. Suppose we have an embedding of our factor  $N$  in a larger factor  $M$ , i.e. we have a subfactor  $N \subset M$ . Let  $\iota : N \hookrightarrow M$  be the inclusion map which we may consider as an  $M$ - $N$  morphism. Choose a representative  $\bar{\iota} : M \rightarrow N$  of the conjugate  $N$ - $M$  sector. Then  $\theta = \bar{\iota}\iota$  is Longo's dual canonical endomorphism, and we call  $N \subset M$  a braided subfactor if its sector  $[\theta]$  decomposes exclusively into subsectors of our braided system  ${}_N\mathcal{X}_N$ , i.e. if

$$[\theta] = \bigoplus_{\rho \in {}_N\mathcal{X}_N} n_\rho [\rho], \quad n_\rho = \langle \rho, \theta \rangle \in \{0, 1, 2, \dots\}.$$

It is straightforward to extend a braiding  $\varepsilon$  on  ${}_N\mathcal{X}_N$  to the set  $\Sigma({}_N\mathcal{X}_N)$  of all to equivalent morphisms and direct sums (see e.g. [10]). Then one can define the  $\alpha$ -induced morphisms  $\alpha_\lambda^\pm \in \text{End}(M)$  for  $\lambda \in \Sigma({}_N\mathcal{X}_N)$  by the Longo-Rehren formula [45] which concretely realizes a “cohomological extension” suggested by Roberts [58] about 24 years ago. Namely one puts

$$\alpha_\lambda^\pm = \bar{\iota}^{-1} \circ \text{Ad}(\varepsilon^\pm(\lambda, \theta)) \circ \lambda \circ \bar{\iota}.$$

Then  $\alpha_\lambda^+$  and  $\alpha_\lambda^-$  extend  $\lambda$ , i.e.  $\alpha_\lambda^\pm \circ \iota = \iota \circ \lambda$ , which in turn implies  $d_{\alpha_\lambda^\pm} = d_\lambda$  by the multiplicativity of the minimal index [44]. Let  $\gamma = \iota\bar{\iota}$  denote Longo's canonical endomorphism [42] from  $M$  into  $N$ . Then there is an isometry  $v \in \text{Hom}(\text{id}, \gamma)$  such that any  $m \in M$  is uniquely decomposed as  $m = nv$  with  $n \in N$  [43]. Thus the action of the extensions  $\alpha_\lambda^\pm$  are uniquely characterized by the relation  $\alpha_\lambda^\pm(v) = \varepsilon^\pm(\lambda, \theta)^* v$  which can be derived from the braiding fusion equations Eq. (2.1). We have  $\alpha_\lambda^\pm$  is a conjugate for  $\alpha_\lambda^\pm$ , moreover  $\alpha_{\lambda\mu}^\pm = \alpha_\lambda^\pm \alpha_\mu^\pm$  if also  $\mu \in \Sigma({}_N\mathcal{X}_N)$ , and clearly  $\alpha_{\text{id}_N}^\pm = \text{id}_M$  (proofs can be found in [5], and for a similar framework – the relations are explained in [67] – in the earlier work [66]). In general one has

$$\text{Hom}(\lambda, \mu) \subset \text{Hom}(\alpha_\lambda^\pm, \alpha_\mu^\pm) \subset \text{Hom}(\iota\lambda, \iota\mu), \quad \lambda, \mu \in \Sigma({}_N\mathcal{X}_N).$$

Now let us count the common subsectors of  $\alpha$ -induced morphisms with different chirality “+” and “−” by defining a “coupling matrix”  $Z$  with entries

$$Z_{\lambda,\mu} = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle, \quad \lambda, \mu \in {}_N\mathcal{X}_N. \quad (2.4)$$

Clearly,  $Z_{\lambda,\mu}$  are non-negative integers since dimensions of vector spaces. Moreover,  $Z_{0,0} = 1$  due to  $\alpha_{\text{id}_N}^\pm = \text{id}_M$ . Also note that Eq. (2.4) immediately yields Eq. (1.6). It has been shown in [10, Thm. 5.7] that in fact

$$YZ = ZY, \quad \Omega Z = Z\Omega,$$

no matter whether the braiding is degenerate or not. So here we have a notion of modular invariants arising from subfactors which extends the non-degenerate (i.e. modular) case to non-unitary S-matrices. We now would like to see more structure and to understand e.g. the connection between modular invariants and graphs or the Moore-Seiberg machinery involving fusion rules automorphism invariants, type I and type II invariants etc. from the operator algebraic point of view. As we shall see, this viewpoint opens up new insights and resolves so far somewhat mysterious phenomena. For that, we have to analyze the structure of  $\alpha$ -induced sectors.

### 3 Structure of modular invariants from subfactors: Induced sector systems, fusion and graphs

Let  ${}_M\mathcal{X}_M \subset \text{End}(M)$  denote a system of endomorphisms consisting of a choice of representative endomorphisms of each irreducible subsector of sectors of the form  $[\iota\lambda\bar{\iota}]$ ,  $\lambda \in {}_N\mathcal{X}_N$ . We choose  $\text{id} \in \text{End}(M)$  representing the trivial sector in  ${}_M\mathcal{X}_M$ .



Then we define similarly the “chiral” systems  ${}_M\mathcal{X}_M^\pm$  and the “ $\alpha$ -system”  ${}_M\mathcal{X}_M^\alpha$  to be the subsystems of endomorphisms  $\beta \in {}_M\mathcal{X}_M$  such that  $[\beta]$  is a subsector of  $[\alpha_\lambda^\pm]$  and of  $[\alpha_\lambda^+ \alpha_\mu^-]$ , respectively, with  $\lambda, \mu \in {}_N\mathcal{X}_N$  varying. (Any subsector of  $[\alpha_\lambda^+ \alpha_\mu^-]$  is automatically a subsector of  $[\nu\bar{\nu}]$  for some  $\nu \in {}_N\mathcal{X}_N$ .) The “neutral” (or “ambichiral”) system is defined as the intersection  ${}_M\mathcal{X}_M^0 = {}_M\mathcal{X}_M^+ \cap {}_M\mathcal{X}_M^-$ , so that  ${}_M\mathcal{X}_M^0 \subset {}_M\mathcal{X}_M^\pm \subset {}_M\mathcal{X}_M^\alpha \subset {}_M\mathcal{X}_M$ . Their “global indices”, i.e. their sums over the squares of the statistical dimensions are denoted by  $w_0, w_\pm, w_\alpha$  and  $w$ , and thus fulfill  $1 \leq w_0 \leq w_\pm \leq w_\alpha \leq w$ . It turns out that the relative sizes of these systems, which are measured by such global indices, are completely encoded in the coupling matrix  $Z$ , namely [11, Prop. 3.1]

$$w_+ = \frac{w}{\sum_{\lambda \in {}_N\mathcal{X}_N} d_\lambda Z_{\lambda,0}} = \frac{w}{\sum_{\lambda \in {}_N\mathcal{X}_N} Z_{0,\lambda} d_\lambda} = w_-, \quad (3.1)$$

and [8, Prop. 3.1]

$$w_\alpha = \frac{w}{\sum_{\lambda \in {}_N\mathcal{X}_N^{\text{deg}}} Z_{0,\lambda} d_\lambda}, \quad w_0 = \frac{w_+^2}{w_\alpha}, \quad (3.2)$$

where  ${}_N\mathcal{X}_N^{\text{deg}} \subset {}_N\mathcal{X}_N$  denotes the subsystem of degenerate morphisms. As a corollary of Eq. (3.2) one obtains that non-degeneracy of the braiding (i.e.  ${}_N\mathcal{X}_N^{\text{deg}} = \{\text{id}\}$ ) implies the “generating property”  ${}_M\mathcal{X}_M^\alpha = {}_M\mathcal{X}_M$ .

Though  ${}_N\mathcal{X}_N$  is braided by assumption, the systems  ${}_M\mathcal{X}_M^\pm$  or  ${}_M\mathcal{X}_M$  will in general not be. In fact, as constructed explicitly in [7], there is only a relative braiding between  ${}_M\mathcal{X}_M^+$  and  ${}_M\mathcal{X}_M^-$ , and this restricts to a proper braiding on the intersection  ${}_M\mathcal{X}_M^0$ . However, the systems  ${}_M\mathcal{X}_M^\pm$  can even be non-commutative. A criterion was found for the case of a non-degenerate braiding: Namely, if we consider the fusion rules of  ${}_M\mathcal{X}_M^\pm$  as finite-dimensional  $C^*$ -algebras  $\text{Furu}({}_M\mathcal{X}_M^\pm)$ , then we have [11, Thm. 4.11]

$$\text{Furu}({}_M\mathcal{X}_M^\pm) \simeq \bigoplus_{\lambda \in {}_N\mathcal{X}_N} \bigoplus_{\tau \in {}_M\mathcal{X}_M^0} \text{Mat}(b_{\tau,\lambda}^\pm) \quad (3.3)$$

with “chiral branching coefficients”  $b_{\tau,\lambda}^\pm = \langle \tau, \alpha_\lambda^\pm \rangle$ . The analogous result for  ${}_M\mathcal{X}_M$  reads [10, Thm. 6.8] (also provided that the braiding on  ${}_N\mathcal{X}_N$  is non-degenerate)

$$\text{Furu}({}_M\mathcal{X}_M) \simeq \bigoplus_{\lambda, \mu \in {}_N\mathcal{X}_N} \text{Mat}(Z_{\lambda,\mu}). \quad (3.4)$$

(The latter decomposition was claimed in a similar form in [48] in the context of Goodman-de la Harpe-Jones subfactors related to  $SU(2)_k$  modular invariants.) Equivalently one may determine the irreducible decomposition of the “regular representations”  $\pi_{\text{reg}}^\pm, \pi_{\text{reg}}$  of the fusion rule algebras  $\text{Furu}({}_M\mathcal{X}_M^\pm), \text{Furu}({}_M\mathcal{X}_M)$ , respectively, and the corresponding irreducible decompositions then read

$$\begin{aligned} \pi_{\text{reg}}^\pm &\simeq \bigoplus_{\lambda \in {}_N\mathcal{X}_N} \bigoplus_{\tau \in {}_M\mathcal{X}_M^0} b_{\tau,\lambda}^\pm \pi_{\tau,\lambda}^\pm, \\ \pi_{\text{reg}} &\simeq \bigoplus_{\lambda, \mu \in {}_N\mathcal{X}_N} Z_{\lambda,\mu} \pi_{\lambda,\mu}, \end{aligned} \quad (3.5)$$

with multiplicities given by the dimensions of the irreducible representations. Note that this is essentially the block-diagonalization of the fusion matrices of the intrinsic fusion rules of the systems  ${}_M\mathcal{X}_M^\pm$  and  ${}_M\mathcal{X}_M$ . Reflecting commutativity properties

of the induced sectors  $[\alpha_\nu^+]$  ( $[\alpha_\nu^-]$ ), these are scalars in the irreducible representations  $\pi_{\tau,\lambda}^+$  ( $\pi_{\tau,\mu}^-$ ) and  $\pi_{\rho,\sigma}$ , given by  $S_{\nu,\lambda}/S_{0,\lambda}$  ( $S_{\nu,\mu}/S_{0,\mu}$ ) and  $S_{\nu,\rho}/S_{0,\rho}$  ( $S_{\nu,\sigma}/S_{0,\sigma}$ ), respectively [11, Cor. 4.15]. However, we can now easily see that the entire system  ${}_M\mathcal{X}_M$  has non-commutative fusion if and only if an entry of  $Z$  is strictly larger than one. There are lots of examples, the simplest given by the  $D_{\text{even}}$  series of  $SU(2)_k$ . Similarly we see that the chiral system  ${}_M\mathcal{X}_M^\pm$  have non-commutative fusion if and only if there is a chiral branching coefficient  $b_{\tau,\lambda}^\pm$  strictly larger than one. To find examples one has to dig a bit further. In fact this happens for a series of conformal inclusions  $SU(n)_n \subset SO(n^2 - 1)_1$  for  $n \geq 4$ , and indeed a non-commutative chiral fusion structure was found for the case  $n = 4$  by direct computation in [66] (see also [6] for a treatment using the Longo-Rehren  $\alpha$ -induction).

So how can we interpret non-commutative fusion rules? Why is there a relative braiding between the possibly non-commutative chiral systems  ${}_M\mathcal{X}_M^+$  and  ${}_M\mathcal{X}_M^-$ ? For this we should think of our endomorphisms in  ${}_N\mathcal{X}_N$  again as DHR endomorphisms of a whole net of algebras  $\{N(I)\}$  over the punctured circle rather than of a single local algebra  $N(I_0)$ . In this context the (subsectors of the)  $\alpha$ -induced sectors are in fact solitonic, i.e. left or right half-line localized depending on their  $\pm$ -chirality, and their respective localization regions intersect exactly on the chosen interval  $I_0$  (see [45]). Then in the DHR framework such solitonic or “twisted” sectors of different chirality can be “pulled apart” and commuted, a procedure which provides in fact unitaries obeying partial braiding properties. However, for sectors with the same half-line localization (or even without any localization as is the case for sectors in the mixed system  ${}_M\mathcal{X}_M^\alpha$ ) this procedure does not work, and so there is no reason why such sectors should have commutative fusion. The neutral system  ${}_M\mathcal{X}_M^0$  however corresponds to proper DHR endomorphisms, and their DHR statistics operators are precisely their restricted relative braiding [7, Prop. 3.15].

But let us return to the regular representations in Eq. (3.5). The representation theoretic point of view has the advantage that we can also consider the left multiplication of  $M$ - $M$  sectors on  $M$ - $N$  sectors: Let  ${}_M\mathcal{X}_N$  denote a system consisting of a choice of representative  $M$ - $N$  morphisms of irreducible subsectors of sectors  $[\iota\lambda]$ , with  $\lambda \in {}_N\mathcal{X}_N$  varying. Then  ${}_M\mathcal{X}_N$  has no intrinsic fusion structure, nevertheless we can consider the representation  $\varrho$  of  $\text{Furu}({}_M\mathcal{X}_M)$  arising from multiplication of  ${}_M\mathcal{X}_M$  on  ${}_M\mathcal{X}_N$ . Its representation matrices  $\varrho(\beta)$ ,  $\beta \in {}_M\mathcal{X}_M$  are given by

$$\varrho(\beta)_{a,b} = \langle b, \beta a \rangle, \quad a, b \in {}_M\mathcal{X}_N.$$

The irreducible decomposition of  $\varrho$  has been determined in [10, Thm. 6.12] to be

$$\varrho \simeq \bigoplus_{\lambda \in {}_N\mathcal{X}_N} \pi_{\lambda,\lambda}. \quad (3.6)$$

Now consider the following matrix representation of  $\text{Furu}({}_N\mathcal{X}_N)$ , with representation matrices  $G_\lambda$ ,  $\lambda \in {}_N\mathcal{X}_N$ , with non-negative integer entries

$$(G_\lambda)_{a,b} = \langle b, \alpha_\lambda^\pm a \rangle, \quad a, b \in {}_M\mathcal{X}_N, \quad (3.7)$$

i.e.  $G_\lambda = \sum_{\beta \in {}_M\mathcal{X}_M^\pm} \langle \beta, \alpha_\lambda^\pm \rangle \varrho(\beta)$ . (It does not depend on the choice of  $\pm$ -chirality.) Thanks to Eq. (3.6), we now know the complete diagonalization of the  $G_\lambda$ ’s: The eigenvalues are of the form  $S_{\lambda,\rho}/S_{0,\rho}$ , and the multiplicities are given by the dimensions  $\dim(\pi_{\rho,\rho}) = Z_{\rho,\rho}$ , the diagonal entries of the modular invariant [11, Thm. 4.16]. So here we have obtained systematically a nimrep of the original Verlinde fusion algebra  $\text{Furu}({}_N\mathcal{X}_N)$  from a subfactor, and it has spectrum canonically associated with the diagonal part of the corresponding modular invariant. Trivially, the

non-negative integer valued matrices can be read as adjacency matrices of graphs and this way we obtain in particular graphs associated to some modular invariant.

Let us take a closer look at the  $M$ - $N$  system  ${}_M\mathcal{X}_N$ . For each  $a \in {}_M\mathcal{X}_N$  we can choose a conjugate  $N$ - $M$  morphism  $\bar{a}$ , and this way we obtain a system  ${}_N\mathcal{X}_M$  of irreducibles  $N$ - $M$  morphisms such that in particular  $\bar{a} \in {}_N\mathcal{X}_M$ . Now for any such  $\bar{a} \in {}_N\mathcal{X}_M$  there is an irreducible subfactor  $\bar{a}(M) \subset N$  and we can form the Jones basic extension

$$\bar{a}(M) \subset N \subset M_a.$$

For the injection homomorphism  $\iota_a : N \hookrightarrow M_a$ , we can choose a conjugate  $M_a$ - $N$  morphism  $\bar{\iota}_a$  such that  $\bar{\iota}_a(M_a) = \bar{a}(M)$ . Hence we have an isomorphism  $\varphi_a : M_a \rightarrow M$  given by  $\varphi_a = \bar{a}^{-1} \circ \bar{\iota}_a$  with inverse  $\varphi_a^{-1} = \bar{\iota}_a^{-1} \circ \bar{a}$ . Note that the dual canonical endomorphism  $\theta_a = \bar{\iota}_a \iota_a$  of the subfactor  $N \subset M_a$  can also be written as  $\theta_a = \bar{a}a$ , and that the Jones index is  $[M_a : N] = d_{\theta_a} = d_a^2$ . Next we consider  $\alpha$ -induction of  $\lambda \in \Sigma({}_N\mathcal{X}_N)$  for  $N \subset M_a$ :

$$\alpha_{a;\lambda}^\pm = \bar{\iota}_a^{-1} \circ \text{Ad}(\varepsilon^\pm(\lambda, \theta_a) \circ \lambda \circ \bar{\iota}_a).$$

It follows from [10, Props. 3.1, 3.3] that  $\varepsilon^\pm(\lambda, \theta_a) \equiv \varepsilon^\pm(\lambda, \bar{a}a)$  can be written as  $\varepsilon^\pm(\lambda, \theta_a) = \bar{a}(U_\lambda^\pm)u_\lambda^\pm$  with unitaries  $U_\lambda^\pm \in \text{Hom}(\alpha_\lambda^\pm a, a\lambda)$  and  $u_\lambda^\pm \in \text{Hom}(\lambda\bar{a}, \bar{a}\alpha_\lambda^\pm)$ . Therefore we find

$$\varphi_a \circ \alpha_{a;\lambda}^\pm \circ \varphi_a^{-1} = \bar{a}^{-1} \circ \text{Ad}(\bar{a}(U_\lambda^\pm)u_\lambda^\pm) \circ \lambda \circ \bar{a} = \text{Ad}(U_\lambda^\pm) \circ \alpha_\lambda^\pm,$$

and consequently maps  $\text{Hom}(\alpha_\lambda^+, \alpha_\mu^\pm) \rightarrow \text{Hom}(\alpha_{a;\lambda}^+, \alpha_{a;\mu}^\pm)$ ,  $t \mapsto \varphi_a^{-1}(U_\mu^\pm t(U_\lambda^+)^*)$  are isomorphisms. In particular, the coupling matrix arising from  $N \subset M_a$  is the same as we obtained from  $N \subset M$ . So here we have found some redundancy for modular invariants from subfactors: Different subfactors can produce the same coupling matrix  $Z$ , and if we start with a given braided subfactor  $N \subset M$ , then we obtain an irreducible subfactor for each morphism in  ${}_N\mathcal{X}_M$  producing the same  $Z$ , though their Jones indices may well be different. The simplest example is the trivial subfactor  $N = M$ , where we obtain Jones extensions  $\lambda(N) \subset N \subset M_\lambda$  for each  $\lambda \in {}_N\mathcal{X}_N$ , and they all will give us the trivial modular invariant  $Z_{\lambda,\mu} = \delta_{\lambda,\mu}$  (cf. [11, Subsect. 6.2]).

It is instructive to use these observations to demonstrate that braided subfactors corresponding to the  $SU(2)_k$  systems can only produce modular invariant coupling matrices with diagonal entries given as Coxeter exponents of A-D-E Dynkin diagrams — even if we would not know anything about the list of modular invariants [12, 13, 38]. So suppose we have a given braided subfactor  $N \subset M$  for the system  ${}_N\mathcal{X}_N$  corresponding to the  $LSU(2)$  loop group model at a level  $k = 1, 2, 3, \dots$ , so that in particular the endomorphisms are labelled by spins  $j = 0, 1, 2, \dots, k$ , we have fusion rules

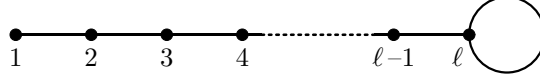
$$N_{j,j'}^{j''} = \begin{cases} 1 & |j - j'| \leq j'' \leq \min(j + j', 2k - j - j'), \\ 0 & \text{otherwise,} \end{cases} \quad j + j' + j'' \in 2\mathbb{Z},$$

and the statistics phases are given by

$$\omega_j = \exp(2\pi i h_j), \quad h_j = \frac{j(j+2)}{4k+8}.$$

Now consider the (adjacency matrix of the)  $M$ - $N$  fusion graph  $G_1$ , i.e. the matrix of Eq. (3.7) corresponding to the spin  $j = 1$ , and let us wonder how it might look like. Since we deal with the generator we will obtain a connected graph, and since  $\alpha$ -induction preserves statistical dimensions, we know already that its largest

eigenvalue only be  $d_1 = 2 \cos(\pi/(k+2)) < 2$ . This already forces  $G_1$  to be one of the A-D-E Dynkin diagrams with dual Coxeter number  $h = k+2$  or if  $k = 2\ell - 1$  is odd it could also be a tadpole  $T_\ell$  which is displayed in Fig. 1 (see e.g. [15] for such arguments). The eigenvalues of  $T_\ell$  are all simple and given by  $2 \cos((r+1)\pi/(k+2))$



**Figure 1** Tadpole graph  $T_\ell$ ,  $\ell = 2, 3, 4, \dots$

with (“exponents”)  $r = 0, 2, 4, \dots, k-1$ . So if we pretend not to know that there is no modular invariant at level  $k = 2\ell - 1$  with  $Z_{j,j} = 1$  if  $j$  is even and  $Z_{j,j} = 0$  if  $j$  is odd, then we have to rule out the possibility that there is a subfactor with  $T_\ell$  as  $M$ - $N$  fusion graph  $G_1$ . This is easy: Assume that there is a subfactor  $N \subset M$  which produces a coupling matrix with these diagonal entries. Note that  $T_\ell$  is  $A_{2\ell}/\mathbb{Z}_2$  without a fixed point, and thus the Perron-Frobenius weights of each vertex of  $T_\ell$  which is labelled by  $j+1$  in Fig. 1 is the same as for the unfolded  $A_{2\ell}$ , i.e.  $\sin((j+1)\pi/(k+2))/\sin(\pi/(k+2))$ . These numbers must be the statistical dimensions, up to an overall normalizing factor which is fixed by the condition that the global index of  ${}_M\mathcal{X}_N$  is the same as  $w$ , the global index of  ${}_N\mathcal{X}_N$  (see e.g. [10, p. 465]). It is then easy to see that the statistical dimension of the  $M$ - $N$  morphism which corresponds to the extremal vertex labelled by “1” in Fig. 1 is  $\sqrt{2}$ . By the above arguments, there must hence be subfactor which produces the same coupling matrix and which has index two, i.e. is the unique  $N \subset N \rtimes \mathbb{Z}_2$ . This would imply that there is an automorphism  $\sigma \in {}_N\mathcal{X}_N$  such that  $\sigma^2 = \text{id}$  and  $[\sigma] \neq [\text{id}]$ . But the only non-trivial automorphism in the  $N$ - $N$  system is the spin  $j = k$  simple current, however, by Rehren’s lemma [53, Lemma 4.4.] this one cannot fulfill  $\sigma^2 = \text{id}$  because its conformal dimension  $k/4$  does not give a statistics phase which is a second root of unity — contradiction.

#### 4 Structure of modular invariants from subfactors: Type I coupling matrices and fusion rule isomorphisms

Next we turn to the discussion of the distinction of type I and type II invariants in the subfactor framework. In our general setting we have

$$Z_{\lambda,\mu} = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle = \sum_{\tau \in {}_M\mathcal{X}_M^0} b_{\tau,\lambda}^+ b_{\tau,\mu}^-,$$

with chiral branching coefficients  $b_{\tau,\lambda}^\pm = \langle \tau, \alpha_\lambda^\pm \rangle$ . To get this in the form of Eq. (1.7) we would need  $b_{\tau,\lambda}^- = b_{\vartheta(\tau),\lambda}^+$  for a permutation of the extended system, being identified as the neutral system  ${}_M\mathcal{X}_M^0$ . Note that by  $\vartheta(0) = 0$  and  $b_{\tau,0}^\pm = \delta_{\tau,0}$  (do not worry that we denote both the original and the extended “vacuum” i.e. identity morphism by the same symbol “0”) we are automatically forced to have symmetric vacuum coupling  $Z_{\lambda,0} = Z_{0,\lambda}$ . This corresponds basically to an “extension of the chiral algebras by primary fields”, namely those which appear in the vacuum column or equivalently in the vacuum row of the coupling matrix  $Z$ . However, there are more general cases as we shall see, which correspond to different extensions for the left and right chiral algebra, and then the vacuum column of the coupling matrix will be different from its vacuum row. In this case we will be lead to different

labelling sets of extended sectors so that the extended modular invariant coupling matrix is

$$Z_{\tau_+, \tau_-}^{\text{ext}} = \delta_{\tau_+, \vartheta(\tau_-)},$$

where  $\vartheta$  is now an isomorphism between the two sets of extended fusion rules, still subject to  $\vartheta(0) = 0$ . Note that when we have two different labelling sets it makes no sense to ask whether a coupling matrix is symmetric or not.

In the subfactor context, the crucial condition for type I coupling matrices turns out to be the “chiral locality condition”, namely the following eigenvalue condition for the statistics operator  $\varepsilon^+(\theta, \theta)$ :

$$\varepsilon^+(\theta, \theta)\gamma(v) = \gamma(v) \quad (4.1)$$

The name chiral locality was given since it was shown in [45] that Eq. (4.1) is a necessary and sufficient condition for locality of the extended net of (here: chiral) observables in the nets of subfactors framework. When chiral locality does hold then [7, Prop. 3.3]

$$\langle \alpha_\lambda^\pm, \beta \rangle = \langle \lambda, \sigma_\beta \rangle,$$

whenever  $\beta \in {}_M\mathcal{X}_M^\pm$ . In particular, when  $\beta = \tau$  is neutral, i.e. lies in the intersection  ${}_M\mathcal{X}_M^0 = {}_M\mathcal{X}_M^+ \cap {}_M\mathcal{X}_M^-$ , then

$$b_{\tau, \lambda}^+ = b_{\tau, \lambda}^- \equiv b_{\tau, \lambda},$$

and we have a block decomposition or “type I” modular invariant

$$Z_{\lambda, \mu} = \sum_{\tau \in {}_M\mathcal{X}_M^0} b_{\tau, \lambda} b_{\tau, \mu}.$$

Now let us characterize to the other extreme case in the subfactor context: pure permutation invariants. As a corollary of Eq. (3.1), the following conditions are equivalent [11, Prop. 3.2]:

- we have  $Z_{\lambda, 0} = \delta_{\lambda, 0}$ ,
- we have  $Z_{0, \lambda} = \delta_{\lambda, 0}$ ,
- we have  ${}_M\mathcal{X}_M^0 = {}_M\mathcal{X}_M$ ,
- The coupling matrix  $Z$  is a permutation, fixing the vacuum and corresponding to a fusion rule automorphism.

For the general case we would like to decompose a modular invariant into its two parts, a type I part together with a twist, and in order to take care of heterotic vacuum coupling we will need to implement such a twist by an isomorphism rather than an automorphism. First we characterize chiral locality. Indeed the following conditions are equivalent [8, Prop. 3.2]:

- We have  $Z_{\lambda, 0} = \langle \theta, \lambda \rangle$  for all  $\lambda \in {}_N\mathcal{X}_N$ .
- We have  $Z_{0, \lambda} = \langle \theta, \lambda \rangle$  for all  $\lambda \in {}_N\mathcal{X}_N$ .
- Chiral locality holds:  $\varepsilon^+(\theta, \theta)v^2 = v^2$ .

In other words: chiral locality holds if and only if the dual canonical endomorphism is entirely “visible” in the vacuum row (and hence column) of the coupling matrix. Using results on intermediate subfactors from [35], we showed the following for a braided subfactor  $N \subset M$  producing a coupling matrix  $Z$  (no matter whether the braiding is non-degenerate or not): There are always [8, Thm. 4.7] intermediate subfactors  $N \subset M_\pm \subset M$ , where  $N \subset M_+$  and  $N \subset M_-$  fulfill the chiral locality condition and produce coupling matrices (“type I parents”)  $Z^+$  and  $Z^-$ , respectively, such that  $Z_{\lambda, 0} = Z_{\lambda, 0}^+ = Z_{0, \lambda}^+$ ,  $Z_{0, \lambda} = Z_{\lambda, 0}^- = Z_{0, \lambda}^-$ . Moreover [8, Thm. 5.5],

the neutral systems arising from  $N \subset M$  and  $N \subset M_{\pm}$  are canonically isomorphic, and whenever  $M_+ = M_-$  then we have in fact  $Z_{\lambda,\mu}^+ = \sum_{\tau} b_{\tau,\lambda} b_{\tau,\mu} = Z_{\lambda,\mu}^-$  and  $Z_{\lambda,\mu} = \sum_{\tau} b_{\tau,\lambda} b_{\vartheta(\tau),\mu}$  is of Moore-Seiberg [46], Dijkgraaf-Verlinde [18] form, where now the fusion rule automorphism arises from the isomorphic neutral systems. It is however important to notice that  $M_+ \neq M_-$ , even  $Z^+ \neq Z^-$  and  $Z_{\lambda,0} \neq Z_{0,\lambda}$  can occur. In fact we realized in [8] the following coupling matrix for  $SO(16)_1$  (actually  $SO(n)_1$  with  $n$  any multiple of 16) from some subfactor  $N \subset M$ :

$$Z = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The intermediate subfactors  $N \subset M_+$  and  $N \subset M_-$  produce type I parent coupling matrices

$$Z^+ = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Z^- = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

respectively. (The first labels for rows and columns always correspond to the vacuum.) Because these invariants can be realized from subfactors these coupling matrices are not spurious in the sense that there are 2D RCFT's such that they encode the coupling of left and right chiral sectors — more about this in the following section.

So what is the physical significance of the intermediate factors  $M_+$  and  $M_-$ ? This can be understood if one considers certain “canonical tensor product subfactors”  $N \otimes N^{\text{opp}} \subset B$  which are directly related to the possible existence of some 2D RCFT containing chiral subtheories described by  $N$  and encoded in the coupling matrix  $Z$  [55, 56]. Then  $M_+ \otimes M_-^{\text{opp}} \subset B$  turns out to be intermediate, and in the physical interpretation of [55],  $M_+$  and  $M_-$  correspond precisely to the maximally extended chiral algebras (in a sensible meaning). For more details, see [8].

## 5 On the existence of 2D RCFT's and the realization of modular invariants from subfactors

The situation for modular invariants from subfactors can simply be stated as follows: For a given type III von Neumann factor  $N$  equipped with a braided system of endomorphism  ${}_N\mathcal{X}_N$ , any embedding  $N \subset M$  of  $N$  in a larger factor  $M$  which is compatible with the system  ${}_N\mathcal{X}_N$  (in the sense that the dual canonical endomorphism decomposes in  ${}_N\mathcal{X}_N$ ) defines a coupling matrix  $Z$  through  $\alpha$ -induction. This matrix  $Z$  commutes with the matrices  $Y$  and  $\Omega$  arising from the braiding and in turn is a “modular invariant” whenever the braiding is non-degenerate. Suppose we start with a system corresponding to some known RCFT data. More concretely, let us consider the situation that our factor  $N$  arises as a local factor  $N = N(I_{\circ})$  of a conformally (here: Möbius) covariant net  $\{N(I)\}$  over  $\mathbb{R}$  (or equivalently  $S^1 \setminus \zeta$ ), as for example of the above sketched  $SU(n)_k$  loop group model. Then the following questions are natural:

**Problem 5.1** Is any coupling matrix which can be produced by some braided extension  $N \subset M$  a physical invariant?

And conversely:

**Problem 5.2** Can any physical modular invariant be realized from some braided extension  $N \subset M$ ?

The first difficulty here is that one needs to specify what the term “physical” means. Quite often in the literature, any matrix commuting with  $S$  and  $T$  and subject to the constraint that all entries are non-negative integers and with normalization  $Z_{0,0} = 1$  is called a physical invariant. Well, with this interpretation of “physical” the solution of Problem 5.1 is trivially the answer “Yes” since we have already established that our coupling matrices have these properties. It is also not too difficult to see that the solution of Problem 5.2 is just “No” with this interpretation of “physical”: Namely, our general theory says that there is always some associated extended theory carrying another representation of the modular group  $SL(2; \mathbb{Z})$  which is compatible with the chiral branching rules (see [8, Sect. 6]). It is however known [60, 63, 24] that there are “spurious” modular invariants satisfying the above constraints but which do not admit an extended modular S-matrix.

Another, physically much more interesting specification of “physical” (but unfortunately mathematically harder to reach) is that  $Z$  arises from “the existence of some 2D RCFT”. A reasonable way of making this precise seems for us to be the concept of chiral observables as light-cone nets built in an observable net over 2D Minkowski space [55]. And in fact Rehren has shown [56] that in the above situation any braided extension  $N \subset M$  determines an entire local 2D conformal field theory over Minkowski space, and that indeed the vacuum Hilbert space of the 2D net decomposes upon restriction to the tensor product of the left and right chiral observables according to Eq. (1.3) with  $Z$  being precisely the matrix arising from  $N \subset M$  through  $\alpha$ -induction. Therefore we obtain a positive solution for Problem 5.1 even with this more subtle notion of a “physical invariant”.

Now let us turn to the converse direction, Problem 5.2. It should be noticed that there can be local 2D extensions of tensored left and right chiral observables (besides the trivial extension) which are completely compatible with conformal symmetry and whose coupling matrices do commute with  $T$  (due to locality) but are not S-invariant (see Rehren’s contribution to this volume). However, Problem 5.2 is restricted to modular invariant coupling matrices, and considering T- and S-invariant 2D extensions, we tend to believe that the solution to Problem 5.2 is again a “Yes”. However, this requires a proof! But since the general classification problem for modular invariants is not solved in general and since it is still a quite subtle question to distinguish physical from spurious invariants even if somebody provides you with a complete list of normalized, non-negative integer matrices in the commutant of  $S$  and  $T$  for some given model, for the time being the state of the art seems to admit only the following recipe:

1. Pick the first  $Z$  from the list and
  - (a) either realize it from a subfactor (and possibly classify inequivalent realizations),
  - (b) or disprove the existence of a subfactor producing this coupling matrix.
2. Pick the next  $Z$ .
3. If you are lucky, do these steps for entire classes of  $Z$ ’s rather than for single ones, so that you can cover certain parts of any list.

Tackling step 1.(a) with a case-by-case analysis has been carried out for a few models. A relation between the A-D-E modular invariants of  $SU(2)$  and A-D-E Goodman-de la Harpe-Jones subfactors [31] can be found in [48]. The type III

subfactors for  $E_6$ ,  $E_8$  and  $D_4$  where analyzed in [66] and found to produce these Dynkin diagrams as fusion diagrams of sectors. The subfactors for the entire  $D_{\text{even}}$  series were constructed in [6] and shown to produce the diagrams as fusion graphs, and that the modular invariants can be recovered by the formula Eq. (2.4) was proven in [7]. The remaining cases were treated in the  $\alpha$ -induction setting extensively in [11]. For  $SU(3)$ , the type I exceptional cases were first analysed in the subfactor context in [66], further analysis also covering the  $\mathcal{D}$  series was carried out in [6, 7, 11], and Ocneanu claimed the solution of the  $SU(3)$  problem in January 2000.

Anyway, let us try to tackle step 1.(a) in the more efficient way, namely looking at classes of modular invariants. Our discussion will be focussed on  $SU(n)_k$ , but the general arguments can also be translated to other models. First of all the trivial invariants,  $Z_{\lambda,\mu} = \delta_{\lambda,\mu}$ , are obtained from the trivial subfactor  $N \subset M$  with  $M = N$ . Next, there are the exceptional modular invariants arising from conformal inclusions. A conformal inclusion means that the level 1 representations of some loop group of a Lie group restrict in a finite manner to the positive energy representations of a certain embedded loop group of an embedded (simple) Lie group at some level. For  $SU(2)$ , the modular invariants arising from conformal embeddings are the  $E_6$ ,  $E_8$  and  $D_4$  ones, corresponding to embeddings  $SU(2)_{10} \subset SO(5)_1$ ,  $SU(2)_{28} \subset (G_2)_1$  and  $SU(2)_4 \subset SU(3)_1$ , respectively. (The latter happens to be a simple current invariant at the same time.) For  $SU(3)$ , the invariants from conformal embeddings are  $\mathcal{D}^{(6)}$ ,  $\mathcal{E}^{(8)}$ ,  $\mathcal{E}^{(12)}$  and  $\mathcal{E}^{(24)}$ , corresponding to  $SU(3)_3 \subset SO(8)_1$ ,  $SU(3)_5 \subset SU(6)_1$ ,  $SU(3)_9 \subset (E_6)_1$ ,  $SU(3)_{21} \subset (E_7)_1$ , respectively. By taking such an embedding as a local subfactor in the vacuum representation, any conformal inclusion determines a braided subfactor of finite index (see [64, 54, 57, 45, 66, 6]), which in turn produces a modular invariant, being precisely the type I (since the embedding level one theory is always local) exceptional invariant which arises from the diagonal invariant of the extended theory [7]. So here a class of exceptional modular invariants is covered in the subfactor context at one stroke, and the consequently existing extended RCFT is of course the level 1 theory of the larger affine Lie algebra.

The situation is even better for simple current invariants, which in a sense produce the majority of non-trivial modular invariants. Simple currents [59] are primary fields with unit quantum dimension and appear in our framework automorphisms in the system  ${}_N\mathcal{X}_N$ . They form a closed abelian group  $G$  under fusion which is hence a product of cyclic groups. Simple currents give rise to modular invariants, and all such invariants have been classified [30, 40]. As focussing on  $SU(n)$  here, we will simply consider cyclic simple current groups  $\mathbb{Z}_n$ .

By taking a generator  $[\sigma]$  for of the cyclic simple current group  $\mathbb{Z}_n$  we can construct the crossed product subfactor  $N \subset M = N \rtimes \mathbb{Z}_n$  whenever we can choose a representative  $\sigma$  in each such simple current sector such that we have exact cyclicity  $\sigma^n = \text{id}$  (and not only as sectors). As we are starting with a chiral quantum field theory, Rehren's lemma [53] applies which states that such a choice is possible if and only if the statistics phase  $\omega_\sigma$  is an  $n$ -th root of unity, i.e. if and only if the conformal weight  $h_\sigma$  is an integer multiple of  $1/n$ . Sometimes this may only be possible for a simple current subgroup  $\mathbb{Z}_m \subset \mathbb{Z}_n$  (with  $m$  a divisor of  $n$ ) but any such non-trivial ( $m \neq 1$ ) subgroup gives rise to a non-trivial subfactor and in turn to a modular invariant. It is easy to see that in fact all simple current invariants are realized this way. For  $SU(n)_k$  the simple current group  $\mathbb{Z}_n$  corresponds to weights  $k\Lambda_{(j)}$ ,  $j = 0, 1, \dots, n-1$ . The conformal dimensions are  $h_{k\Lambda_{(j)}} = kj(n-j)/2n$ , which



by Rehren's Lemma [53] allow for full  $\mathbb{Z}_n$  extensions except when  $n$  is even and  $k$  is odd in which case the maximal extension is  $N \subset M = N \rtimes \mathbb{Z}_{n/2}$  because we can only use the even labels  $j$ . (This reflects the fact that e.g. for  $SU(2)$  there are no D-invariants at odd levels.) Thus Rehren's lemma has told us that extensions are labelled by all the divisors of  $n$  unless  $n$  is even and  $k$  is odd in which case they are labelled by the divisors of  $n/2$ . This matches exactly the simple current modular invariant classification of [30, 40]. An extension by a simple current subgroup  $\mathbb{Z}_m$ , with  $m$  is a divisor of  $n$  or  $n/2$ , is moreover local, if the generating current (and hence all in the  $\mathbb{Z}_m$  subgroup) has integer conformal weight,  $h_{k\Lambda(q)} \in \mathbb{Z}$ , where  $n = mq$ . This happens exactly if  $kq \in 2m\mathbb{Z}$  if  $n$  is even, or  $kq \in m\mathbb{Z}$  if  $n$  is odd [7]. For  $SU(2)$  this corresponds to the  $D_{\text{even}}$  series whereas the  $D_{\text{odd}}$  series are non-local extensions. For  $SU(3)$ , there is a simple current extension at each level, but only those at  $k \in 3\mathbb{Z}$  are local. Clearly, the cases with chiral locality match exactly the type I simple current modular invariants.

With these techniques we can obtain a large number of modular invariants from subfactors. Nevertheless we still do not have a systematic procedure to get *all* physical invariants. The more problematic cases are typically the exceptional type II invariants. Therefore let us now tackle a large class of exceptional type II invariants, namely those which are type II descendants of conformal embeddings.

## 6 Type II descendants of modular invariants from conformal inclusions

Since for any conformal inclusion the ambient theory is described by the level 1 representations of the embedding loop group and therefore is typically a pure simple current theory (whenever simply laced Lie groups are dealt with), the subfactors producing their modular invariants can be constructed by simple current methods. Therefore we will obtain the relevant subfactors for type II descendant modular invariants, e.g. the conjugation  $CZ$  of a conformal inclusion invariant  $Z$  through crossed products.

For a while we will be looking at the so-called  $\mathbb{Z}_n$  conformal field theories as treated in [14], which have  $n$  sectors, labelled by  $j = 0, 1, 2, \dots, n-1 \pmod{n}$ , obeying  $\mathbb{Z}_n$  fusion rules, and conformal dimensions of the form  $h_j = aj^2/2n \pmod{1}$ , where  $a$  is an integer mod  $2n$ ,  $a$  and  $n$  coprime and  $a$  is even whenever  $n$  is odd. The modular invariants of such models have been classified [14]. They are labelled by the divisors  $\delta$  of  $\tilde{n}$ , where  $\tilde{n} = n$  if  $n$  is odd and  $\tilde{n} = n/2$  if  $n$  is even. Explicitly, the modular invariants  $Z^{(\delta)}$  are given by

$$Z_{j,j'}^{(\delta)} = \begin{cases} 1 & \text{if } j, j' = 0 \pmod{\alpha} \text{ and } j' = \omega(\delta)j \pmod{n/\alpha}, \\ 0 & \text{otherwise,} \end{cases}$$

where  $\alpha = \gcd(\delta, \tilde{n}/\delta)$  so that there are numbers  $r, s \in \mathbb{Z}$  such that  $r\tilde{n}/\delta\alpha - s\delta/\alpha = 1$  and then  $\omega(\delta)$  is defined as  $\omega(\delta) = r\tilde{n}/\delta\alpha + s\delta/\alpha$ . The trivial invariant corresponds to  $\delta = \tilde{n}$ , i.e.  $Z^{(\tilde{n})} = \mathbf{1}$  and  $\delta = 1$  gives the charge conjugation matrix,  $Z^{(1)} = C$ .

It is straightforward combinatorics [9] to show that

$$Z_{j,j}^{(\delta)} = \begin{cases} 1 & \text{if } j = 0 \pmod{\tilde{n}/\delta}, \\ 0 & \text{otherwise.} \end{cases}$$

This yields  $\text{tr}(Z^{(\delta)}) = \epsilon\delta$  for the trace of  $Z^{(\delta)}$ ; here  $\epsilon = 2$  if  $n$  is even and  $\epsilon = 1$  if  $n$  is odd. Now suppose that for such a  $\mathbb{Z}_n$  theory at hand we have corresponding braided automorphisms  $\tau_j$  of some type III factor  $N$ , obeying  $\mathbb{Z}_n$  fusion rules and such that their statistical phases are given by  $e^{2\pi i h_j}$  with conformal weights  $h_j$  as above (as

is the case e.g. for  $SU(n)$  level 1 theories). Note that if  $n$  is odd then we can always assume that  $\tau_1^n = \text{id}$  as morphisms (and our system can be chosen as  $\{\tau_1^j\}_{j=0}^{n-1}$ ). However, if  $n$  is even, then we cannot choose a representative of the sector  $[\tau_1]$  such that its  $n$ -th power gives the identity, nevertheless we can always assume that  $\tau_\epsilon^{\tilde{n}} = \text{id}$ . Thus we have a simple current (sub-) group  $\mathbb{Z}_{\tilde{n}}$ , for which we can form the crossed product subfactor  $N \subset M = N \rtimes_{\mathbb{Z}_{\tilde{n}/\delta}} N$  for any divisor  $\delta$  of  $\tilde{n}$ . It is quite easy to see that  $N \subset M = N \rtimes_{\tau_\epsilon \delta} \mathbb{Z}_{\tilde{n}/\delta}$  indeed realizes  $Z^{(\delta)}$ : The crossed product by  $\mathbb{Z}_{\tilde{n}/\delta}$  gives the dual canonical endomorphism sector  $[\theta] = [\text{id}] \oplus [\tau_{\epsilon\delta}] \oplus [\tau_{\epsilon\delta}^2] \oplus \dots \oplus [\tau_{\epsilon\delta}^{\tilde{n}/\delta-1}]$ . The formula  $\langle \iota\tau_j, \iota\tau_{j'} \rangle = \langle \theta\tau_j, \tau_{j'} \rangle$  then shows that the system of  $M$ - $N$  morphisms is labelled by  $\mathbb{Z}_n/\mathbb{Z}_{\tilde{n}/\delta} \simeq \mathbb{Z}_{\epsilon\delta}$ , i.e.  $\#_M \mathcal{X}_N = \epsilon\delta$ . Therefore our general theory implies that the modular invariant arising from  $N \subset M = N \rtimes_{\mathbb{Z}_{\tilde{n}/\delta}} N$  has trace equal to  $\epsilon\delta$ , and thus must be  $Z^{(\delta)}$ . Consequently all modular invariants classified in [14] are realized from subfactors.

It is instructive to apply the above results to descendant modular invariants of conformal inclusions. Let us consider the conformal inclusion  $SU(4)_6 \subset SU(10)_1$ . The associated modular invariant, which can be found in [59], reads

$$\mathcal{Z} = \sum_{j \in \mathbb{Z}_{10}} |\chi^j|^2$$

with  $SU(10)_1$  characters decomposing into  $SU(4)_6$  characters as

$$\begin{aligned} \chi^0 &= \chi_{0,0,0} + \chi_{0,6,0} + \chi_{2,0,2} + \chi_{2,2,2}, & \chi^5 &= \chi_{0,0,6} + \chi_{6,0,0} + \chi_{0,2,2} + \chi_{2,2,0}, \\ \chi^1 &= \chi_{0,0,2} + \chi_{2,4,0} + \chi_{2,1,2}, & \chi^6 &= \chi_{4,0,0} + \chi_{0,2,4} + \chi_{1,2,1}, \\ \chi^2 &= \chi_{0,1,2} + \chi_{2,3,0} + \chi_{3,0,3}, & \chi^7 &= \chi_{3,0,1} + \chi_{1,2,3} + \chi_{0,3,0}, \\ \chi^3 &= \chi_{1,0,3} + \chi_{3,2,1} + \chi_{0,3,0}, & \chi^8 &= \chi_{0,3,2} + \chi_{2,1,0} + \chi_{3,0,3}, \\ \chi^4 &= \chi_{0,0,4} + \chi_{4,2,0} + \chi_{1,2,1}, & \chi^9 &= \chi_{2,0,0} + \chi_{0,4,2} + \chi_{2,1,2}. \end{aligned}$$

We observe that  $Z$  has 32 diagonal entries. As usual, this invariant can be realized from the conformal inclusion subfactor

$$\pi^0(L_I SU(4))'' \subset \pi^0(L_I SU(10))'',$$

with  $\pi^0$  denoting the level 1 vacuum representation of  $LSU(10)$ . We will denote this subfactor by  $N \subset M_+$ . The dual canonical endomorphism sector corresponds to the vacuum block:

$$[\theta_+] = [\lambda_{0,0,0}] \oplus [\lambda_{0,6,0}] \oplus [\lambda_{2,0,2}] \oplus [\lambda_{2,2,2}].$$

Proceeding with  $\alpha$ -induction  $\lambda_{p,q,r} \mapsto \alpha_{\pm;p,q,r}^\pm \in \text{End}(M_+)$ , it is a straightforward calculation that the graphs describing left multiplication by fundamental generators  $[\alpha_{\pm;1,0,0}^\pm]$  and  $[\alpha_{\pm;0,1,0}^\pm]$  (which is the same as right multiplication by  $[\lambda_{1,0,0}]$  and  $[\lambda_{0,1,0}]$ , respectively) on the system of  $M_+$ - $N$  sectors gives precisely the graphs found by Petkova and Zuber [50, Figs. 1 and 2] by their more empirical procedure to obtain graphs with spectrum matching the diagonal part of some given modular invariant. In our framework, the graph [50, Fig. 1] obtains the following meaning: Take the outer wreath, pick a vertex with 4-ality 0 and label it by  $[\iota_+] \equiv [\tau_0 \iota_+]$ , where  $\iota_+ : N \hookrightarrow M_+$  denotes the injection homomorphism, as usual. Going around in a counter-clockwise direction the vertices will then be the marked vertices labelled by the  $\mathbb{Z}_{10}$  sectors  $[\tau_1 \iota_+]$ ,  $[\tau_2 \iota_+]$ ,  $\dots$ ,  $[\tau_9 \iota_+]$  of  $SU(10)_1$ . Passing to the next inner wreath the 4-ality 1 vertex adjacent to  $[\iota_+]$  is then the sector  $[\alpha_{\pm;1,0,0}^\pm \iota_+] = [\iota_+ \lambda_{1,0,0}]$ , and the others its  $\mathbb{Z}_{10}$  translates. Similarly the inner wreath consists of the  $\mathbb{Z}_{10}$  translates of  $[\iota_+ \lambda_{0,1,0}]$ . The remaining two vertices in the center correspond to

subsectors of the reducible  $[\iota\lambda_{1,1,0}]$  and  $[\iota\lambda_{0,1,1}]$ . The graph itself then represents left (right) multiplication by  $[\alpha_{\pm,1,0,0}^\pm]$  ( $[\lambda_{1,0,0}]$ ).

As for  $LSU(10)$  at level 1 we are in fact dealing with a  $\mathbb{Z}_n$  conformal field theory, we have  $n = 10$  and  $\tilde{n} = 5$ , thus we know that there are only two modular invariants: The diagonal one which in restriction to  $LSU(4)$  gives exactly the above type I invariant  $Z \equiv Z^{(5)}$ , but there is also the charge conjugation invariant  $CZ \equiv Z^{(1)}$ , written in characters as

$$CZ = \sum_{j \in \mathbb{Z}_{10}} \chi^j (\chi^{-j})^*.$$

Whereas  $Z^{(5)}$  can be thought of as the trivial extension  $M_+ \subset M_+$ , the conjugation invariant  $Z^{(1)}$  can be realized from the crossed product  $M_+ \subset M = M_+ \rtimes \mathbb{Z}_5$  which has dual canonical endomorphism sector

$$[\theta^{\text{ext}}] = [\tau_0] \oplus [\tau_2] \oplus [\tau_4] \oplus [\tau_6] \oplus [\tau_8].$$

So far we have considered the situation on the “extended level”, but we may now descend to the level of  $SU(4)_6$  sectors and characters. Namely we may consider the subfactor  $N \subset M = M_+ \rtimes \mathbb{Z}_5$ . Its dual canonical endomorphism sector  $[\theta]$  is obtained by  $\sigma$ -restriction of  $[\theta^{\text{ext}}]$  which can now be read off from the character decomposition,

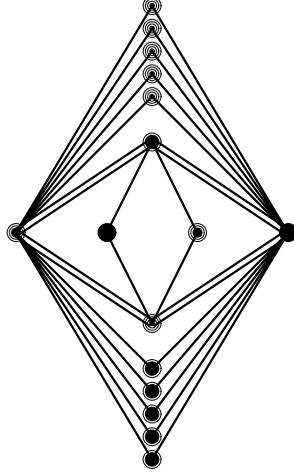
$$\begin{aligned} [\theta] &= \bigoplus_{j=0}^4 [\sigma_{\tau_{2j}}] = [\lambda_{0,0,0}] \oplus [\lambda_{0,6,0}] \oplus [\lambda_{2,0,2}] \oplus [\lambda_{2,2,2}] \oplus [\lambda_{0,1,2}] \oplus [\lambda_{2,3,0}] \\ &\quad \oplus 2[\lambda_{3,0,3}] \oplus [\lambda_{0,0,4}] \oplus [\lambda_{4,2,0}] \oplus 2[\lambda_{1,2,1}] \oplus [\lambda_{4,0,0}] \oplus [\lambda_{0,2,4}] \oplus [\lambda_{0,3,2}] \oplus [\lambda_{2,1,0}]. \end{aligned}$$

This subfactor produces the conjugation invariant  $CZ$  written in  $SU(4)_6$  characters which is the same as taking the original  $SU(4)_6$  conformal inclusion invariant and conjugating on the level of the  $SU(4)_6$  characters. Note that this invariant has only 16 diagonal entries.

When passing from  $M_+$  to  $M = M_+ \rtimes \mathbb{Z}_5$ , the  $M_+$ - $N$  system will change to the  $M$ - $N$  system in such a way that all sectors which are translates by  $\tau_{2j}$ ,  $j = 0, 1, 2, 3, 4$ , have to be identified, and similarly fixed points split. Thus our new system of  $M$ - $N$  morphisms will be some kind of orbifold of the old one. To see this, we first recall that all the irreducible  $M_+$ - $N$  morphisms are of the form  $\beta_{\iota_+}$  with  $\beta \in {}_{M_+}\mathcal{X}_{M_+}^\pm$ . To such an irreducible  $M_+$ - $N$  morphism  $\beta_{\iota_+}$  we can now associate an  $M$ - $N$  morphism  $\iota^{\text{ext}}\beta_{\iota_+}$  which may no longer be irreducible; here  $\iota^{\text{ext}}$  is the injection homomorphism  $M_+ \hookrightarrow M$ . Then the reducibility can be controlled by Frobenius reciprocity as we have

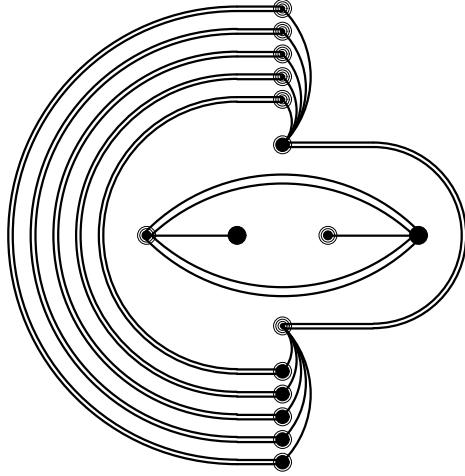
$$\langle \iota^{\text{ext}}\beta_{\iota_+}, \iota^{\text{ext}}\beta'_{\iota_+} \rangle = \langle \theta^{\text{ext}}\beta_{\iota_+}, \beta'_{\iota_+} \rangle,$$

and  $\theta^{\text{ext}} = \bar{\tau}^{\text{ext}}\iota^{\text{ext}}$ . Carrying out the entire computation we find that in contrast to the original 32  $M_+$ - $N$  sectors we are left with only 16  $M$ - $N$  sectors, and the right multiplication by  $[\lambda_{1,0,0}]$  is displayed graphically as in Fig. 2. Here the 4-ality 0, 1, 2, 3 of the vertices are indicated by solid circles of decreasing size. The  $[\iota]$  vertex (with  $\iota = \iota^{\text{ext}}\iota_+$  denoting the injection homomorphism  $N \hookrightarrow M$  of the total subfactor  $N \subset M = M_+ \rtimes \mathbb{Z}_5$ ) is the 4-ality 0 vertex in the center of the picture, and the 4-ality 1 vertex above corresponds to  $[\iota\lambda_{1,0,0}]$ . Each group of five vertices on the top and the bottom of the picture arise from the splitting of the two central vertices of the graphs in [50] as they are  $\mathbb{Z}_5$  fixed points. That our orbifold graph inherits the 4-ality of the original graph is due to the fact that all entries in  $[\theta]$



**Figure 2** Graph  $G_1$  associated to the conjugation invariant of the conformal inclusion  $SU(4)_6 \subset SU(10)_1$

have 4-ality zero which in turn comes from the fact that all even marked vertices (corresponding to the subgroup  $\mathbb{Z}_5 \subset \mathbb{Z}_{10}$ ) of the graph of Petkova and Zuber have 4-ality zero. We also display the graph corresponding to the second fundamental representation, namely the right multiplication by  $[\lambda_{0,1,0}]$  in Fig. 2.



**Figure 3** Graph  $G_2$  associated to the conjugation invariant of the conformal inclusion  $SU(4)_6 \subset SU(10)_1$

Let us finally remark that the conformal inclusion invariant  $Z$  has the funny property  $Z^*Z = 3Z + CZ$ . This is remarkable as this is the first type I invariant we have encountered which does not fulfill  $Z^*Z = x_+Z$ , where  $x_+ = \sum_{\lambda} Z_{\lambda,0}^2$ . Since the diagonal part of  $Z^*Z$  describes the spectral properties of the fusion graphs of chiral generators in the full system [10, 11] we expect that for the conformal inclusion subfactor  $N \subset M_+$  the fusion graph of the chiral generator  $[\alpha_{1,0,0}^+]$  (or

$[\alpha_{1,0,0}^-])$  in the full  $M_+-M_+$  system has four connected components (each of which corresponds to a nimrep), three of them being the graph of [50, Fig. 1] and one component being the graph in Fig. 2. In fact, it is easy to see that for any (non-degenerately) braided subfactor  $N \subset M$  the number of  ${}_M\mathcal{X}_M^0$  fusion orbits in  ${}_M\mathcal{X}_M^\pm$  is equal to the number of  ${}_M\mathcal{X}_M^\mp$  fusion orbits in  ${}_M\mathcal{X}_M$ . Moreover, a simple Perron-Frobenius argument shows that this number is exactly  $x_\pm$  (with  $x_- = \sum_\lambda Z_{0,\lambda}^2$ ).

The conformal inclusion  $SU(3)_5 \subset SU(6)_1$  can be treated along the same lines [9]. The associated  $SU(3)_5$  modular invariant, is labelled by the graph  $\mathcal{E}^{(8)}$ . The ambient  $SU(6)_1$  has besides the diagonal only the conjugation invariant which is the obtained through a  $\mathbb{Z}_3$  extension on top of the conformal inclusion subfactor, and the  $\mathbb{Z}_3$  quotient collapses the 12 vertices of  $\mathcal{E}^{(8)}$  to 4, yielding exactly the graph  $\mathcal{E}^{(8)*}$  in the list of Di Francesco and Zuber (see e.g. [2]). So with this procedure we understand quite generally why the descendants of modular invariants of conformal inclusions (whenever the ambient theory has  $\mathbb{Z}_n$  fusion rules) are in fact labelled by orbifold graphs of the graph labelling the original, block-diagonal conformal inclusion invariant, and why the conjugation invariant corresponds to the maximal  $\mathbb{Z}_{\tilde{n}}$  orbifold.

In the above examples, the trivial and conjugation invariant of the extended theory still remain distinct when written in terms of the  $SU(4)_6$  characters. This need not be the case in general. Let us look at a familiar modular invariant of  $SU(3)$  at level 9, namely

$$\mathcal{Z}_{\mathcal{E}^{(12)}} = |\chi_{0,0} + \chi_{9,0} + \chi_{0,9} + \chi_{4,1} + \chi_{1,4} + \chi_{4,4}|^2 + 2|\chi_{2,2} + \chi_{5,2} + \chi_{2,5}|^2,$$

which arises from the conformal embedding  $SU(3)_9 \subset (E_6)_1$ . Now  $E_6$  at level 1 gives a  $\mathbb{Z}_3$  theory and in terms of the extended characters the above invariant is the trivial extended invariant

$$\mathcal{Z}_{\mathcal{E}_1^{(12)}} = |\chi^0|^2 + |\chi^1|^2 + |\chi^2|^2,$$

using obvious notation. Here both the  $(E_6)_1$  characters  $\chi^1$  and  $\chi^2$  specialize to  $\chi_{2,2} + \chi_{5,2} + \chi_{2,5}$  in terms of  $SU(3)_9$  variables. Let  $N \subset M_+$  denote the conformal inclusion subfactor obtained by analogous means as in the previous example.<sup>3</sup> It has been treated in [66, 7] and produces the graph  $\mathcal{E}_1^{(12)}$  of the list of Di Francesco and Zuber as chiral fusion graphs — and in turn as  $M_+-N$  fusion graph, thanks to chiral locality.

Corresponding to the two divisors 3 and 1 of 3, we know that besides the trivial there is only the conjugation invariant of our  $\mathbb{Z}_3$  theory. It is given as

$$C\mathcal{Z}_{\mathcal{E}_2^{(12)}} = |\chi^0|^2 + \chi^1(\chi^2)^* + \chi^2(\chi^1)^*$$

but this distinct invariant restricts to the same invariant  $\mathcal{Z}_{\mathcal{E}^{(12)}}$  when specialized to  $SU(3)_9$  variables. Nevertheless we will obtain a different subfactor  $N \subset M$  since the conjugation invariant of our  $\mathbb{Z}_3$  theory is realized from the extension

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<sup>3</sup>The notation  $N \subset M_+$  for the conformal inclusion subfactor which indicates that it will be the maximal local intermediate subfactor (à la [8]) of some extension  $N \subset M = M_+ \rtimes \mathbb{Z}_\ell$  using simple currents is appropriate for the examples discussed here but not in general. Other conformal inclusions as for instance  $SU(7)_7 \subset SO(48)_1$  or  $SU(8)_{10} \subset SU(36)_1$  can also have type I descendants as coming from *local* simple current extensions of the ambient theory (e.g.  $SU(36)_1 \rtimes \mathbb{Z}_3$  for the latter example). In other words, for such descendant invariants, the ambient affine Lie algebra does *not* provide the maximally extended chiral algebra which then is actually larger.

$M_+ \subset M = M_+ \rtimes \mathbb{Z}_3$ . In particular, the subfactor  $N \subset M$  has dual canonical endomorphism sector

$$[\theta] = [\lambda_{0,0}] \oplus [\lambda_{9,0}] \oplus [\lambda_{0,9}] \oplus [\lambda_{4,1}] \oplus [\lambda_{1,4}] \oplus [\lambda_{4,4}] \oplus 2[\lambda_{2,2}] \oplus 2[\lambda_{5,2}] \oplus 2[\lambda_{2,5}],$$

determined by  $\sigma$ -restriction of

$$[\theta^{\text{ext}}] = [\tau_0] \oplus [\tau_1] \oplus [\tau_2].$$

As before, the  $M$ - $N$  system can be obtained from the  $M_+$ - $N$  system by dividing out the cyclic symmetry carried by  $[\theta^{\text{ext}}]$ . In terms of graphs, the cyclic  $\mathbb{Z}_3$  symmetry corresponds to the three wings of the graph  $\mathcal{E}_1^{(12)}$  which are transformed into each other by translation through the  $[\tau_j]$ 's, and dividing out this symmetry gives exactly the graph  $\mathcal{E}_2^{(12)}$  as the wings are identified whereas each vertex on the middle axis splits into three nodes of identical Perron-Frobenius weight. This way we understand the graph  $\mathcal{E}_2^{(12)}$  as the label for the conjugation invariant  $Z_{\mathcal{E}_2^{(12)}}$  of  $Z_{\mathcal{E}_1^{(12)}}$  which accidentally happens to be the same as the self-conjugate  $Z_{\mathcal{E}^{(12)}}$  when specialized to  $SU(3)_9$  variables. So here we have found some kind of two-fold degeneracy of the modular invariant  $Z_{\mathcal{E}^{(12)}}$ .

An even higher degeneracy appears for the modular invariant of  $SU(3)_3$  which comes from the conformal embedding  $SU(3)_3 \subset SO(8)_1$ . Let  $N \subset M_+$  be the corresponding local subfactor, as usual. Let us briefly recall some facts about the ambient  $SO(8)_1$  theory. It has four sectors, the basic (0), vector (v), spinor (s) and conjugate spinor (c) module. The conformal dimensions are given as  $h_0 = 0$ ,  $h_v = h_s = h_c = 1/2$ , and the sectors obey  $\mathbb{Z}_2 \times \mathbb{Z}_2$  fusion rules. The Kac-Peterson matrices are given explicitly as

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad T = e^{\pi i/3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

It is easy to see that there are exactly six modular invariants, namely the six permutations of v, s and c. Thus we are exclusively dealing with automorphism invariants here, and in terms of  $SU(3)_3$  variables they all specialize to the same modular invariant

$$\mathcal{Z}_{\mathcal{D}^{(6)}} = |\chi_{0,0} + \chi_{3,0} + \chi_{0,3}|^2 + 3|\chi_{1,1}|^2 \quad (6.1)$$

since the  $SO(8)_1$  characters decompose upon restriction to  $SU(3)$  variables into the level 3 characters as

$$\chi^0 = \chi_{0,0} + \chi_{3,0} + \chi_{0,3}, \quad \chi^v = \chi^s = \chi^c = \chi_{1,1}.$$

The  $\mathbb{Z}_2 \times \mathbb{Z}_2$  fusion rules for  $SO(8)_1$  models were proven in the DHR framework in [4], and together with the conformal spin and statistics theorem [21, 20, 32] we conclude that there is a system  $\{\text{id}, \tau_v, \tau_s, \tau_c\} \subset \text{End}(M_+)$  of braided endomorphisms, such that the statistics S- and T-matrices are given exactly as above. Because the statistics phases are given as  $\omega_v = \omega_s = \omega_c = -1$ , we can assume that the morphisms in the system obey the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  fusion rules even by individual multiplication,

$$\tau_v^2 = \tau_s^2 = \tau_c^2 = \text{id}, \quad \tau_v \tau_s = \tau_s \tau_v = \tau_c,$$

thanks to Rehren's lemma [53]. Hence we can extend  $M_+$  in three ways as crossed products by  $\mathbb{Z}_2$ , and the corresponding dual canonical endomorphism sectors  $[\theta]$  are respectively  $[\text{id}] \oplus [\tau_v]$ ,  $[\text{id}] \oplus [\tau_s]$  and  $[\text{id}] \oplus [\tau_c]$ . We can also extend by the full

group  $\mathbb{Z}_2 \times \mathbb{Z}_2$  giving instead  $[\text{id}] \oplus [\tau_v] \oplus [\tau_s] \oplus [\tau_c]$ . Checking  $\langle \iota\lambda, \iota\mu \rangle = \langle \theta\lambda, \mu \rangle$  for  $\lambda, \mu = \text{id}, \tau_v, \tau_s, \tau_c$ , we find that there are only two  $M$ - $N$  sectors for the  $\mathbb{Z}_2$  extensions and only a single one, namely  $[\iota]$ , for the full  $\mathbb{Z}_2 \times \mathbb{Z}_2$  extension. So  $\text{tr}Z = \#_M \mathcal{X}_N$  tells us that the modular invariants of  $SO(8)_1$  which arise from the  $M_+ \subset M_+ \rtimes \mathbb{Z}_2$  extensions can only be the transpositions whereas the full  $M_+ \subset M_+ \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$  extension must produce one of the two cyclic permutations. In fact it can produce both of them because the matrices are relatively transpose and therefore one can be obtained from the other by exchanging braiding and opposite braiding. It is easy to see that the extension by  $\tau_v$  gives exactly the transposition fixing  $v$  (and analogously for  $s$  and  $c$ ): By  $\langle \alpha_\lambda^+, \alpha_\mu^- \rangle \leq \langle \theta\lambda, \mu \rangle$  we find  $Z_{v,s} = 0 = Z_{v,c}$  for  $[\theta] = [\text{id}] \oplus [\tau_v]$ , leaving only this possibility.

So what does the braided subfactor  $N \subset M$  with  $M$  one of these extensions give? Clearly, they all produce the invariant of Eq. (6.1). What are the relevant  $SU(3)$  graphs? A little calculation shows easily that the  $\mathbb{Z}_2$  extensions give the graph  $\mathcal{D}^{(6)*}$  of the Di Francesco-Zuber list whereas  $N \subset M_+ \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$  yields the same graph as  $N \subset M_+$ , namely  $\mathcal{D}^{(6)}$ . This is one of the examples where different 2D CFT's, namely with different extended coupling matrices (here the trivial one and a non-trivial cyclic permutation of  $v, s, c$ ), are associated with the same graph. Therefore we should not consider graphs as complete labels of 2D extensions of some given chiral algebra.

By the way, a similar phenomenon seems already to happen for the A-D-E invariants of  $SU(2)_k$ . For any  $D_{\text{even}}$  invariant, the exchange of the two (“marked”) vertices sitting on the short legs of the D-graph is a fusion rule automorphism of the neutral system (see [6, Subsect. 3.4]) leaving conformal dimensions invariant, but the exchange is invisible on the level of  $SU(2)_k$  characters since both correspond to the character  $\chi_{k/2}$ . So in this sense, all the  $D_{\text{even}}$  invariants are two-fold degenerate. A special case is  $D_{10}$  where one can also permute these two vertices with the third vertex on the longest leg. So here acts<sup>4</sup> again the permutation group  $S_3$ . But the four permutations involving this vertex are visible on the level of  $SU(2)_{16}$  characters (since  $\chi_{k/2}$  and  $\chi_2$  are different), and they indeed produce the  $E_7$  invariant which is henceforth four-fold degenerate. The  $A_{k+1}$ ,  $E_6$  and  $E_8$  invariants do not have a degeneracy.

## References

- Baumgärtel, H.: *Operatoralgebraic methods in quantum field theory*. Berlin: Akademie Verlag, 1995
- Behrend, R.E., Pearce, P.A., Petkova, V.B., Zuber, J.-B.: *Boundary conditions in rational conformal field theories*. Nucl. Phys. **B570** (2000), 525-589
- Böckenhauer, J.: *Localized endomorphisms of the chiral Ising model*. Commun. Math. Phys. **177** (1996), 265-304
- Böckenhauer, J.: *An algebraic formulation of level one Wess-Zumino-Witten models*. Rev. Math. Phys. **8** (1996), 925-947
- Böckenhauer, J., Evans, D.E.: *Modular invariants, graphs and  $\alpha$ -induction for nets of subfactors. I*. Commun. Math. Phys. **197** (1998), 361-386
- Böckenhauer, J., Evans, D.E.: *Modular invariants, graphs and  $\alpha$ -induction for nets of subfactors. II*. Commun. Math. Phys. **200** (1999), 57-103
- Böckenhauer, J., Evans, D.E.: *Modular invariants, graphs and  $\alpha$ -induction for nets of subfactors. III*. Commun. Math. Phys. **205** (1999), 183-228
- Böckenhauer, J., Evans, D.E.: *Modular invariants from subfactors: Type I coupling matrices and intermediate subfactors*. Preprint math.OA/9911239, to appear in Commun. Math. Phys.

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<sup>4</sup>This was pointed out to us by K.-H. Rehren.

- Böckenhauer, J., Evans, D.E.: *Modular invariants from subfactors*. Preprint based on lectures given by D.E. Evans at “Quantum symmetries in theoretical physics and mathematics” in Bariloche, Argentina, math.OA/0006114
- Böckenhauer, J., Evans, D.E., Kawahigashi, Y.: *On  $\alpha$ -induction, chiral generators and modular invariants for subfactors*. Commun. Math. Phys. **208** (1999), 429-487
- Böckenhauer, J., Evans, D.E., Kawahigashi, Y.: *Chiral structure of modular invariants for subfactors*. Commun. Math. Phys. **210** (2000), 733-784
- Cappelli, A., Itzykson, C., Zuber, J.-B.: *Modular invariant partition functions in two dimensions*. Nucl. Phys. **B280** (1987), 445-465
- Cappelli, A., Itzykson, C., Zuber, J.-B.: *The A-D-E classification of minimal and  $A_1^{(1)}$  conformal invariant theories*. Commun. Math. Phys. **113** (1987), 1-26
- Degiovanni, P.: *Z/NZ Conformal field theories*. Commun. Math. Phys. **127** (1990), 71-99
- Di Francesco, P.: *Integrable lattice models, graphs and modular invariant conformal field theories*. Int. J. Mod. Phys. **A7** (1992), 407-500
- Di Francesco, P., Zuber, J.-B.:  *$SU(N)$  lattice integrable models associated with graphs*. Nucl. Phys. **B338** (1990), 602-646
- Di Francesco, P., Zuber, J.-B.:  *$SU(N)$  lattice integrable models and modular invariance*. In: *Recent Developments in Conformal Field Theories*. Trieste 1989, Singapore: World Scientific, 1990, pp. 179-215
- Dijkgraaf, R., Verlinde, E.: *Modular invariance and the fusion algebras*. Nucl. Phys. (Proc. Suppl.) **5B** (1988), 87-97
- Evans, D.E., Kawahigashi, Y.: *Quantum symmetries on operator algebras*. Oxford: Oxford University Press, 1998
- Fredenhagen, K., Rehren, K.-H., Schroer, B.: *Superselection sectors with braid group statistics and exchange algebras. II*. Rev. Math. Phys. **Special issue** (1992), 113-157
- Fröhlich, J., Gabbiani, F.: *Braid statistics in local quantum theory*. Rev. Math. Phys. **2** (1990), 251-353
- Fröhlich, J., Gabbiani, F.: *Operator algebras and conformal field theory*. Commun. Math. Phys. **155** (1993), 569-640
- Fuchs, J.: *Affine Lie algebras and quantum groups*. Cambridge: Cambridge University Press, 1992
- Fuchs, J., Schellekens, A.N., Schweigert, C.: *Galois modular invariants of WZW models*. Nucl. Phys. **B437** (1995), 667-694
- Gannon, T.: *WZW commutants, lattices and level-one partition functions*. Nucl. Phys. **B396** (1993), 708-736
- Gannon, T.: *The classification of affine  $SU(3)$  modular invariants*. Commun. Math. Phys. **161** (1994), 233-264
- Gannon, T.: *The level two and three modular invariants of  $SU(n)$* . Lett. Math. Phys. **39** (1997), 289-298
- Gannon, T.: *The Cappelli-Itzykson-Zuber A-D-E classification*. Preprint, math.QA/9902064
- Gannon, T.: *Monstrous moonshine and the classification of CFT*. Lectures given in Istanbul, August 1998, math.QA/9906167
- Gato-Rivera, B., Schellekens A.N.: *Complete classification of modular invariants for RCFT's with a center  $(\mathbb{Z}_p)^k$* . Commun. Math. Phys. **145** (1992), 85-121
- Goodman, F., de la Harpe, P., Jones, V.F.R.: *Coxeter graphs and towers of algebras*. MSRI publications 14, Berlin: Springer, 1989
- Guido, D., Longo, R.: *The conformal spin and statistics theorem*. Commun. Math. Phys. **181** (1996), 11-35
- Haag, R.: *Local Quantum Physics*. Berlin: Springer-Verlag, 1992
- Izumi, M.: *Application of fusion rules to classification of subfactors*. Publ. RIMS, Kyoto Univ. **27** (1991), 953-994
- Izumi, M., Longo, R., Popa, S.: *A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras*. J. Funct. Anal. **155** (1998), 25-63
- Jones, V.F.R.: *Index for subfactors*. Invent. Math. **72** (1983), 1-25
- Kač, V.G.: *Infinite dimensional Lie algebras*, 3rd edition, Cambridge: Cambridge University Press, 1990
- Kato, A.: *Classification of modular invariant partition functions in two dimensions*. Mod. Phys. Lett. **A2** (1987), 585-600



- Kosaki, H.: *Extension of Jones theory on Index to arbitrary factors*. J. Funct. Anal. **66** (1986), 123-140
- Kreuzer, M., Schellekens A.N.: *Simple currents versus orbifolds with discrete torsion – a complete classification*. Nucl. Phys. **B411** (1994), 97-121
- Loke, T.: *Operator algebras and conformal field theory of the discrete series representations of  $\text{Diff}(S^1)$* . Dissertation, Cambridge, 1994
- Longo, R.: *Solution of the factorial Stone-Weierstrass conjecture*. Invent. Math. **76** (1984), 145-155
- Longo, R.: *Index of subfactors and statistics of quantum fields II*. Commun. Math. Phys. **130** (1990), 285-309
- Longo, R.: *Minimal index of braided subfactors*. J. Funct. Anal. **109** (1991), 98-112
- Longo, R., Rehren, K.-H.: *Nets of subfactors*. Rev. Math. Phys. **7** (1995), 567-597
- Moore, G., Seiberg, N.: *Naturality in conformal field theory*. Nucl. Phys. **B313** (1989), 16-40
- Nahm, W.: *Lie group exponents and  $SU(2)$  current algebras*. Commun. Math. Phys. **118** (1988), 171-176
- Ocneanu, A.: *Paths on Coxeter diagrams: From Platonic solids and singularities to minimal models and subfactors*. (Notes recorded by S. Goto) In: Rajarama Bhat, B.V. et al. (eds.), Lectures on operator theory, The Fields Institute Monographs, Providence, Rhode Island: AMS publications 2000, pp. 243-323
- Petkova, V.B., Zuber, J.-B.: *From CFT to graphs*. Nucl. Phys. **B463** (1996), 161-193
- Petkova, V.B., Zuber, J.-B.: *Conformal field theory and graphs*. In: Proceedings Goslar 1996 “Group 21”, hep-th/9701103
- Pressley, A., Segal, G.: *Loop groups*. Oxford: Oxford University Press, 1986
- Rehren, K.-H.: *Braid group statistics and their superselection rules*. In: Kastler, D. (ed.): *The algebraic theory of superselection sectors*. Palermo 1989, Singapore: World Scientific, 1990, pp. 333-355
- Rehren, K.-H.: *Space-time fields and exchange fields*. Commun. Math. Phys. **132** (1990), 461-483
- Rehren, K.-H.: *Subfactors and coset models*. In: Dobrev, V. et al (eds.): *Generalized symmetries in physics*. Singapore: World Scientific, 1994, pp. 338-356
- Rehren, K.-H.: *Chiral observables and modular invariants*. Commun. Math. Phys. **208** (2000), 689-712
- Rehren, K.-H.: *Canonical tensor product subfactors*. Commun. Math. Phys. **211** (2000), 395-406
- Rehren, K.-H., Stanev, Y.S., Todorov, I.T.: *Characterizing invariants for local extensions of current algebras*. Commun. Math. Phys. **174** (1996), 605-633
- Roberts, J.E.: *Local cohomology and superselection structure*. Commun. Math. Phys. **51** (1976), 107-119
- Schellekens A.N., Yankielowicz, S.: *Extended chiral algebras and modular invariant partition functions*. Nucl. Phys. **B327** (1989), 673-703
- Schellekens A.N., Yankielowicz, S.: *Field identification fixed points in the coset construction*. Nucl. Phys. **B334** (1990), 67-102
- Toledano Laredo, V.: *Fusion of positive energy representations of  $L\text{Spin}_{2n}$* . PhD Thesis, Cambridge, 1997
- Verlinde, E.: *Fusion rules and modular transformations in 2D conformal field theory*. Nucl. Phys. **B300** (1988), 360-376
- Verstegen, D.: *New exceptional modular invariant partition functions for simple Kac-Moody algebras*. Nucl. Phys. **B346** (1990), 349-386
- Wassermann, A.: *Subfactors arising from positive energy representations of some infinite dimensional groups*. Unpublished notes, 1990
- Wassermann, A.: *Operator algebras and conformal field theory III: Fusion of positive energy representations of  $LSU(N)$  using bounded operators*. Invent. Math. **133** (1998), 467-538
- Xu, F.: *New braided endomorphisms from conformal inclusions*. Commun. Math. Phys. **192** (1998), 347-403
- Xu, F.: *3-Manifold invariants from cosets*. Preprint math.GT/9907077